Partial solutions for the suggested exercises, Section 7.4 (plus a small list for review)

They were for the DGD of May 8th and 10th

(1) Find the following integrals:

$$\int \frac{x^2}{x+1}\, dx.$$

Long division gives you $\int (x - 1 + \frac{1}{x+1})\, dx$.

Integrating, we get $\frac{x^2}{2} - x + \ln(x+1) + C$.

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\, dx.$$

degree of numerator bigger than degree of denominator: we can proceed to partial fractions. Remain to find A, B, C and D:

$$\frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

Solving, we get $A = 3, B = -3, C = 0, D = 2$. Integrating, we get

$$\frac{3}{2}\ln|1+x^2| - 3\arctan(x) + \sqrt{2}\arctan(\frac{x}{\sqrt{2}}) + C$$

$$\int \frac{4x-1}{x^2+x-2}\, dx.$$

Need to factor $x^2 + x - 2 = (x+2)(x-1)$. We can do partial fractions:

$$\frac{4x-1}{(x+2)(x-1)} = \frac{1}{x-1} + \frac{3}{x+2}$$

Integrating, we get

$$\int \frac{4x-1}{x^2+x-2}\, dx = \ln|x-1| + 3\ln|x+2| + C$$

$$\int \frac{6x-5}{2x+3}\, dx.$$

Long division yields

$$3 - \frac{14}{2x+3}$$

Integrating:

$$\int \frac{6x-5}{2x+3}\ dx = 3x - 7\ln|2x+3| + C$$

$$\int \frac{x^2-2x-1}{(x-1)^2(x+1)^2}\ dx = -\frac{1}{2}\ln|x+1| + \frac{1/2}{x+1} + \frac{1}{2}\ln|x-1| + \frac{1/2}{x-1} + C$$

since partial fraction yields

$$\frac{x^2-2x-1}{(x-1)^2(x+1)^2} = \frac{-1/2}{x+1} + \frac{-1/2}{(x+1)^2} + \frac{1/2}{x-1} + \frac{-1/2}{(x-1)^2}$$

$$\int \frac{x^2+1}{x^2-x}\ dx = x - \ln|x| + 2\ln|x-1| + C$$

since long division and partial fractions yields

$$\frac{x^2+1}{x^2-x} = 1 - \frac{1}{x} + \frac{2}{x-1}$$

$$\int \frac{x-3}{(x^2+2x+4)^2}\ dx = \int \frac{x}{(x^2+2x+4)^2}\ dx - 3\int \frac{1}{(x^2+2x+4)^2}\ dx$$

$$= \frac{-\frac{1}{2}}{x^2+2x+4} - 4\int \frac{1}{(x^2+2x+4)^2}\ dx$$

Square completion: $x^2+2x+4 = (x+1)^2+3$ So

$$\int \frac{1}{(x^2+2x+4)^2}\ dx = \int \frac{1}{((x+1)^2+3)^2}\ dx$$

Substituing $\sqrt{3}\tan w = x+1$, $\sqrt{3}\sec^2 w\ dw = dx$ and

$$\int \frac{1}{((x+1)^2+3)^2}\ dx = \int \frac{\sqrt{3}\sec^2 w}{(3\tan^2 w+3)^2}\ dw = \frac{\sqrt{3}}{9}\int \frac{\sec^2 w}{(\tan^2 w+1)^2}\ dw$$

Note that $\tan^2 w + 1 = \sec^2 w$, and that $\sec w = \cos w$, thus

$$\frac{\sqrt{3}}{9}\int \frac{\sec^2 w}{(\tan^2 w+1)^2}\ dw = \frac{\sqrt{3}}{9}\int \cos^2 w\ dw$$

Since $\cos^2 w = \frac{1}{2}(1+\cos(2w))$,

$$\frac{\sqrt{3}}{9}\int \cos^2 w\ dw = \frac{\sqrt{3}}{18}(x + \frac{1}{2}\sin(2w)) = \frac{\sqrt{3}}{18}(x + \frac{1}{2}\sin(2\arctan(\frac{x+1}{\sqrt{3}}))))$$

Summarizing,

$$\int \frac{x-3}{(x^2+2x+4)^2}\ dx = \frac{-\frac{1}{2}}{x^2+2x+4} - 4\left(\frac{\sqrt{3}}{18}\left(x+\frac{1}{2}\sin\left(2\arctan(\frac{x+1}{\sqrt{3}})\right)\right)\right)$$

$$= \frac{-\frac{1}{2}}{x^2+2x+4} - \frac{2\sqrt{3}}{9}x + \frac{\sqrt{3}}{9}\sin\left(2\arctan(\frac{x+1}{\sqrt{3}})\right) + C$$

[Modulo any typos]

$$\int \frac{1}{(x-1)^2(x+4)}\ dx = \frac{1}{25}\ln|x+4| - \frac{1}{25}\ln|x-1| - \frac{\frac{1}{5}}{x-1} + C$$

Since

$$\frac{1}{(x-1)^2(x+4)} = \frac{1/25}{x+4} + \frac{-1/25}{x-1} + \frac{1/5}{(x-1)^2}$$

$$\int \frac{\sin(x)\cos^2(x)}{5+\cos^2(x)}\ dx$$

Substitute $w = \cos x$. Then $dw = -\sin x\ dx$, and

$$\int \frac{\sin(x)\cos^2(x)}{5+\cos^2(x)}\ dx = \int \frac{w^2}{5+w^2}\ dw = \int (1 - \frac{5}{w^2+5})\ dw$$

using long division. Then integrating we get

$$w - \frac{5}{\sqrt{5}}\arctan\left(\frac{w}{\sqrt{5}}\right) = \arccos x - \sqrt{5}\arctan\left(\frac{\arccos x}{\sqrt{5}}\right) + C$$

$$\int \frac{x^3}{(x+1)^3}\ dx = \int (1 - \frac{1}{(x+1)^3} + \frac{3}{(x+1)^2} - \frac{3}{x+1})\ dx$$

by long division and partial fractions. Integrating:

$$= x + \frac{1/2}{(x+1)^2} - \frac{3}{x+1} - 3\ln|x+1| + C$$

$$\int \frac{4x^2+5x+7}{4x^2+4x+5}\ dx = \int (1 + \frac{x+2}{4x^2+4x+5})\ dx$$

by long division. Then splitting the integral and performing square completion on the third part:

$$= \int \left(1 + \frac{1}{8}\frac{8x+4}{4x^2+4x+5} + \frac{3}{2}\frac{1}{(2x+1)^2+4}\right)\ dx$$

First, $\int \frac{8x+4}{4x^2+4x+5}\ dx = \ln|4x^2+4x+5|$.

Then, substituting $2x+1 = 2\tan w$, $dx = \sec^2 w\ dw$, and

$$\int \frac{1}{(2x+1)^2+4}\ dx = \int \frac{\sec^2 w}{4\tan^2 w + 4}\ dw$$

Using trigonometric properties:

$$= \frac{1}{4}\int \frac{\sec^2 w}{\sec^2 w}\ dw = \frac{1}{4}\int 1\ dw = \frac{1}{4}w = \frac{1}{4}\arctan(x + \frac{1}{2})$$

Summarizing:

$$\int \frac{4x^2+5x+7}{4x^2+4x+5}\ dx = \int \left(1 + \frac{1}{8}\frac{8x+4}{4x^2+4x+5} + \frac{3}{2}\frac{1}{(2x+1)^2+4}\right)\ dx$$

$$= x + \frac{1}{8}\ln|4x^2+4x+5| + \frac{3}{8}\arctan(x+\frac{1}{2}) + C$$

(2) Find the following integrals:

$$\#46 \int \frac{dz}{(4-z^2)^{3/2}}$$

Using $z = 2sin(w)$, $dz = 2\cos(w)dw$, you get:

$$\int \frac{dz}{(4-z^2)^{3/2}} = \int \frac{2\cos(w)dw}{(4-4\sin^2 w)^{3/2}} = \int \frac{2\cos(w)dw}{(4\cos^2 w)^{3/2}} = \frac{2}{4^{3/2}}\int \frac{\cos(w)dw}{\cos^3 w}$$

$$= \frac{2}{2^3}\int \sec^2 w\ dw = \frac{1}{2^2}\tan w + C = \frac{1}{4}\tan\arcsin(z/2) + C$$

$$\#45 \int \frac{dt}{t^2\sqrt{1+t^2}}$$

Substitute $t = \tan(z)$, thus $dt = \sec^2 z\ dz$. After some manipulation you get

$$\int \frac{\cos z}{\sin^2 z}\ dz$$

Substitute $u = \sin z$ and you'll find

$$\int \frac{\cos z}{\sin^2 z}\ dz = -\frac{1}{\sin z} = -\frac{1}{\sin\arctan t}$$

$$\#43 \int \frac{x^2}{\sqrt{9-x^2}}\ dx$$

Substitute $z = 3\sin x$.

(3) Show that the following equation holds:

$$\int_{-1}^{1} \frac{dx}{\sqrt{5+2x+x^2}} = \int_{0}^{\frac{\pi}{4}} \sec(w)dw$$

Square completion: $5+2x+x^2 = (x+1)^2 + 2^2$. Then do $x+1 = \tan w$.

(4) Exercise #59:

(a) Show that $\int \frac{1}{\sin^2(x)} dx = -\frac{1}{\tan(x)} + C$.
One method is to use the fund. theorem of calculus: derive $-\frac{1}{\tan(x)}$, you get $\frac{1}{\sin^2(x)}$. Then

$$\int \frac{1}{\sin^2(x)} dx = \int \frac{d\left(-1/\tan(x)\right)}{dx} dx = -\frac{1}{\tan(x)} + C$$

(b) Calculate $\int \frac{dy}{y^2\sqrt{5-y^2}}$.
Do $y = \sqrt{5}\sin x$, you should get back on integral in (a).