hints for the suggested exercises, Section 7.7-8
(Modulo the typos)

(1) Using the comparison test, determine if the following integrals converge or diverge:

$$\#3 \int_1^\infty \frac{x^2+1}{x^3+3x+2}dx$$

Note that when $x \geq 1$, $x^3 \leq x^3+3x+2$ and $x^2 \leq x^2+1$, thus

$$\frac{1}{x} = \frac{x^2}{x^3} \leq \frac{x^2+1}{x^3+3x+2}$$

Since $\int_1^\infty \frac{dx}{x}$ does not converge, so does $\int_1^\infty \frac{x^2+1}{x^3+3x+2}dx$

$$\int_0^\pi \frac{2+\sin\phi}{\phi^2}d\phi$$

Does not converge since $\int_0^\pi \frac{3}{\phi}d\phi$ does not converge (why ?).

$$\#10 \int_{50}^\infty \frac{dz}{z^3}$$

Consider

$$\int_1^\infty \frac{dz}{z^3} = \int_1^{50} \frac{dz}{z^3} + \int_{50}^\infty \frac{dz}{z^3}$$

$$\#8 \int_1^\infty \frac{1}{e^{5t}+2}dt$$

Compare with e^{-5t}

$$\#18 \int_1^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$$

Compare with $1/\sqrt{2\theta^2} = \frac{1}{\theta\sqrt{2}}$ (for which value of θ is this valid ?)

$$\#19 \int_1^\infty \frac{d\theta}{\sqrt{\theta^3+1}}$$

Compare with $1/\sqrt{\theta^3} = 1/\theta^{3/2}$ (For what θ is this true ?)

$$\int_0^{\pi/2} \frac{dx}{x\sin x}$$

Improper only since not defined at 0. around $x = 0$, $\sin x$ behave "like x". Thus we infer this diverges. Between 0 and $\pi/2$, $\sin(x) \leq x$. Thus $x\sin(x) \leq x^2$. Thus

$$\frac{1}{x^2} \leq \frac{1}{x\sin(x)}$$

and the rest follows.

$$\int_0^1 \frac{e^{-x}}{\sqrt{x}}\, dx$$

Problem at 0. From the curve of e^{-x} around $x = 0$ (draw it to see!), we infer that $e^{-x} < 1$ on $[0, 1]$.

CHECK algebraically that it is true.

Then $\frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$, whose integral between 0 and 1 converges. Thus the integral converges.

$$\int_1^\infty \frac{\sin^2 x}{x^2}\, dx$$

Converges, since $\sin^2(x) < 1$.

(2) Determine whether each integral is convergent or divergent. Evaluate those that converge.

$$\int_0^\infty \frac{dt}{(t+2)(t+3)}$$

Only problem at infinity. Guess: converges, since at infinity the denominator acts like t^2. So we can try to evaluate it.

$$\frac{1}{(t+2)(t+3)} = \frac{1}{t+3} - \frac{1}{t+2}$$

So

$$\int_0^b \frac{dt}{(t+2)(t+3)} = \ln 3 - \ln 2 + \ln(b+2) - \ln(b+3)$$

We cannot compute the limit yet, we need to factor:

$$= \ln(\frac{3}{2}) + \ln(\frac{b+2}{b+3})$$

Now, since

$$\lim_{b\to\infty} \frac{b+2}{b+3} = 1$$

the integral converges to $\ln(\frac{3}{2})$.

$$\int_1^\infty \frac{\ln x}{x}\, dx$$

Problem at ∞. Diverges since $\frac{\ln x}{x} > \frac{1}{x}$ when x large enough (for what value ? how to make the comparison test apply ?).

$$\int_{\pi/4}^{\pi/2} \tan^2 w \; dw$$

Problem at $\pi/2$. Cannot really compare easily. However, remember that $\int(1+\tan^2(x))dx = tan(x) + C$ (why ?). Thus

$$\int_{\pi/4}^{b} \tan^2 w \; dw + \int_{\pi/4}^{b} 1 \; dw = \tan(b) - \tan(\pi/4) = \tan(b) - 1$$

Explain why this is enough to show that the original integral diverges.

$$\int_{0}^{\pi} |\sec x| \; dx$$

splitting in two:

$$\int_{0}^{\pi/2} \sec x \; dx - \int_{\pi/2}^{\pi} \sec x \; dx$$

Consider the first one: $\cos(x)$, around $x = \pi/2$, behave like $\pi/2 - x$.

TO DO: study the function $\pi/2 - x - cos(x)$, on $[0, \pi/2]$. You should get that $cos(x) \leq \pi/2 - x$. Thus that $\sec(x) \geq \frac{1}{\pi/2 - x}$.

Now, you know how to handle $\int_0^{\pi/2} \frac{1}{\pi/2-x} dx$, you can show it diverges. Thus the original integral diverges.

$$\int_{0}^{1} \frac{-\ln x}{\sqrt{x}}$$

Problem at 0. Proceed by integration by part, integrating $-1/\sqrt{x}$ and deriving $\ln(x)$. You should get $-2\ln(x)\sqrt{(x)} + 4\sqrt{(x)}$. Now you can compute the integral asked, you get 4.

$$\int_{\pi/4}^{\pi/2} \sec^2 x \; dx$$

You know the antiderivative of $\sec^2(x)$ (hint: it is $\tan(x) + C$). Thus you can show the integral asked diverges.

$$\int_{0}^{\pi/4} \frac{\cos x}{\sqrt{\sin x}} \; dx$$

Converges to $2^{3/4}$: integrate by substituing $t = sin(x)$.

$$\int_{-\infty}^{\infty} e^{-|x|} \; dx$$

Divide in two for removing the absolute value:

$$= \int_{-\infty}^{0} e^{x}\ dx + \int_{0}^{\infty} e^{-x}\ dx$$

In the first one substitute x for $-t$, and you get

$$2 \int_{0}^{\infty} e^{-x}\ dx$$

which converges to 2.

$$\int_{4}^{5} \frac{dx}{(5-x)^{2/3}}\ dx$$

Substitute $t = 5 - x$, you get

$$= \int_{0}^{1} \frac{dt}{t^{2/3}}\ dt$$

which converges to 3.

(3) For what value of p does the following integrals converge or diverge ?

$$\#30 \int_{2}^{\infty} \frac{dx}{x(\ln x)^{p}} \qquad\qquad \#31 \int_{1}^{2} \frac{dx}{x(\ln x)^{p}}$$

Hint: in both cases apply $t = \ln(x)$ as substitution.

(4) Evaluate the following integrals. You might want to split the domain of integration.

$$\int_{0}^{\infty} \frac{dx}{\sqrt{x}(1+x)}$$

Problem at 0 and at ∞. It is then equal (as long as it converges) to

$$= \int_{0}^{1} \frac{dx}{\sqrt{x}(1+x)} + \int_{1}^{\infty} \frac{dx}{\sqrt{x}(1+x)}$$

To solve those, just use the substitution $t = \sqrt{x}$. Both sub-integrals give you $\pi/2$. The final result is then π.

$$\int_{2}^{\infty} \frac{dx}{x\sqrt{x^2-4}}$$

Here, problem at 2 and ∞:

$$= \int_{2}^{4} \frac{dx}{x\sqrt{x^2-4}} + \int_{4}^{\infty} \frac{dx}{x\sqrt{x^2-4}}$$

Here, apply trigonometric substitution $t = 2sin(x)$. The first one gives you $\pi/6$, the second one $\pi/12$. The final answer is then $\pi/4$.