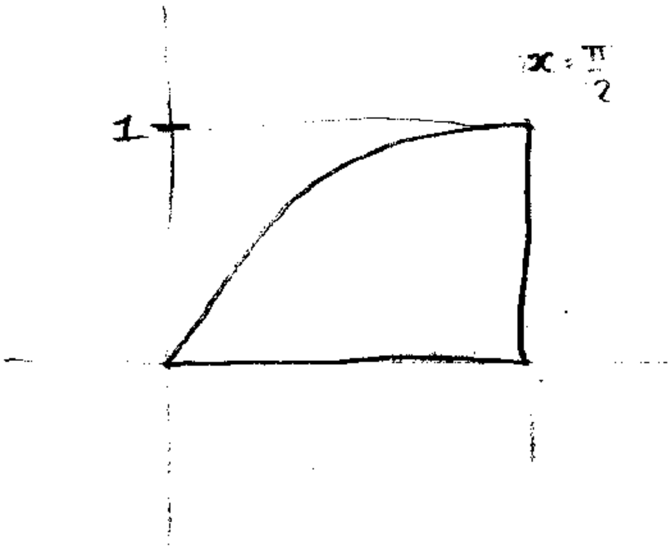
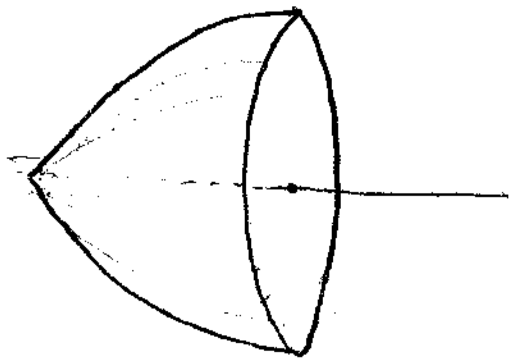
(8) region : $\begin{cases} y = \sin(x) \\ x = \frac{\pi}{2} \\ x\text{-axis} \end{cases}$

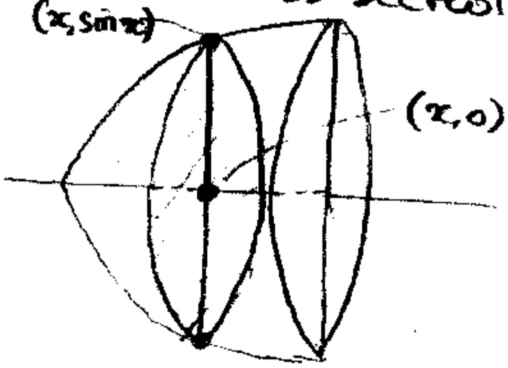


rotated around

(i) the x-axis :



consider a cross-section at x :



it is a circle
of radius $(\sin x)$:
its area is $\pi(\sin x)^2$.

the elementary volume of the slice is $\pi(\sin x)^2 dx$

summing all the slices; we got the volume:

$$\int_0^{\pi/2} \pi(\sin x)^2 dx.$$

$$= \pi\int_0^{\pi/2} (\sin x)^2 dx.$$

by parts: $u = \sin x$ $\quad u' = \cos x$

$v' = \sin x$ $\quad v = -\cos x$

$$= \pi\left[-\sin x \cos x\right]_0^{\pi/2} + \pi\int_0^{\pi/2} \cos^2 x \, dx$$

$$= \pi\Big(\underbrace{-\sin\frac{\pi}{2}}_{=1}\underbrace{\cos\frac{\pi}{2}}_{=0} + \underbrace{\sin 0}_{=0}\,\underbrace{\cos 0}_{=1}\Big) + \pi\int_0^{\pi/2}(1-\sin^2 x)\,dx$$

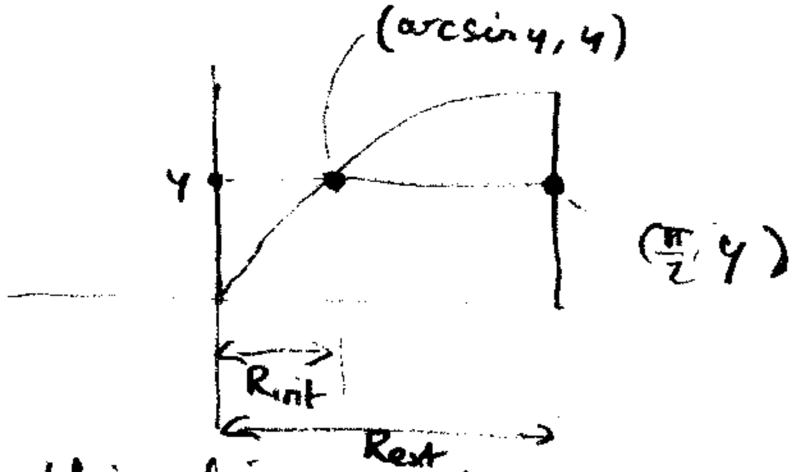
$$= \pi\int_0^{\pi/2} dx \boxed{-\pi\int_0^{\pi/2}\sin^2 x \, dx}$$

thus

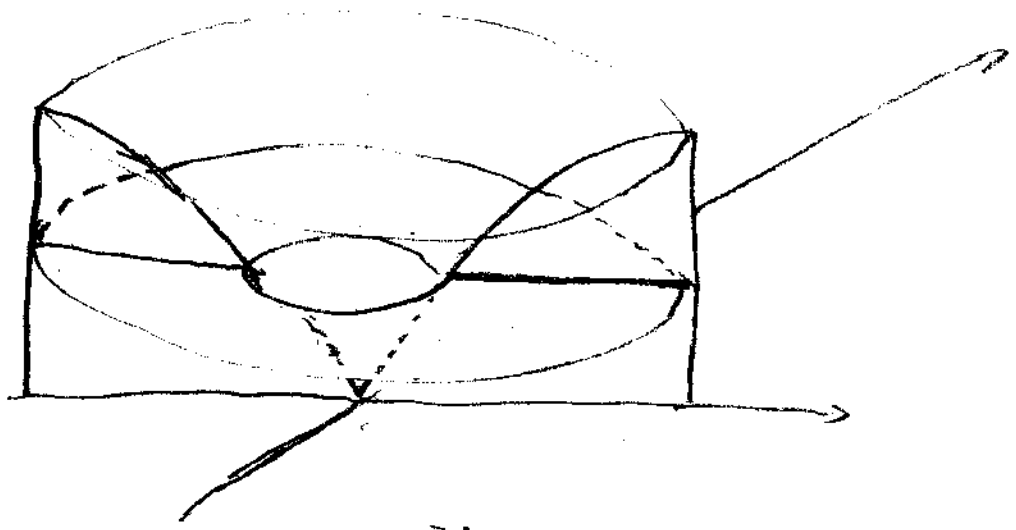
$$\int_0^{\pi/2}\sin^2 x \, dx = \frac{1}{2\pi}\cdot\pi\int_0^{\pi/2} dx$$

$$= \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{\pi}{4}$$

(ii) rotated around the y-axis:



this time we take a slice perp. to the y-axis; at height y: we get an annulus:



interior radius is arcsin y
ext radius is $\frac{\pi}{2}$.

→ area of the ring: $\pi\left(\left(\frac{\pi}{2}\right)^2-(\arcsin y)^2\right)$

Thus the volume of the solid of revolution is : $(y \in [0,1])$

$$\pi\int_0^1 \left(\frac{\pi}{2}\right)^2 - (\arcsin y)^2\, dy$$

$$= \pi\left[\frac{\pi^2}{4}y\right]_0^1 - \pi\int_0^1 (\arcsin y)^2\, dy$$

$$= \frac{\pi^3}{4} - \pi\int_0^1 (\arcsin y)^2\, dy$$

---

let's compute $\int \arcsin^2 y\, dy$:

recall from first week's DGD that

$\int \arcsin y\, dy$ $\quad u = \arcsin y \quad u' = \frac{1}{\sqrt{1-y^2}}$

$\quad v' = 1 \quad v = y$

$= y \arcsin y$

$\quad - \int \frac{y\, dy}{\sqrt{1-y^2}}$

$= y \arcsin y + \sqrt{1-y^2}$

Thus we proceed by parts :

$\int \arcsin^2 y\, dy$ $\quad u = \arcsin y \quad u' = \frac{1}{\sqrt{1-y^2}}$

$\quad v' = \arcsin y \quad v = y\arcsin y + \sqrt{1-y^2}$

$$= y\arcsin^2 y + \arcsin y\sqrt{1-y^2} - \int \frac{y \arcsin y}{\sqrt{1-y^2}}\, dy - \int dy$$

remains $-\int \frac{y \arcsin y}{\sqrt{1-y^2}}\,dy$:

by parts again :

$u = \arcsin y \qquad u' = \frac{1}{\sqrt{1-y^2}}$

$v' = \frac{-y}{\sqrt{1-y^2}} \qquad v = \sqrt{1-y^2}$

$$= \arcsin y\sqrt{1-y^2} - \int dy = \arcsin y\sqrt{1-y^2} - y$$

thus, summarizing :

$$\int \arcsin^2 y\,dy = y\arcsin^2 y + \arcsin y\sqrt{1-y^2} + \underbrace{\int \frac{-y\arcsin y}{\sqrt{1-y^2}}\,dy}_{\arcsin y\sqrt{1-y^2} - y} - \underbrace{\int dy}_{=y}$$

$$\boxed{\int \arcsin^2 y = y\arcsin^2 y + 2\arcsin y\sqrt{1-y^2} - 2y}$$

---

coming back to the volume of the solid, it was :

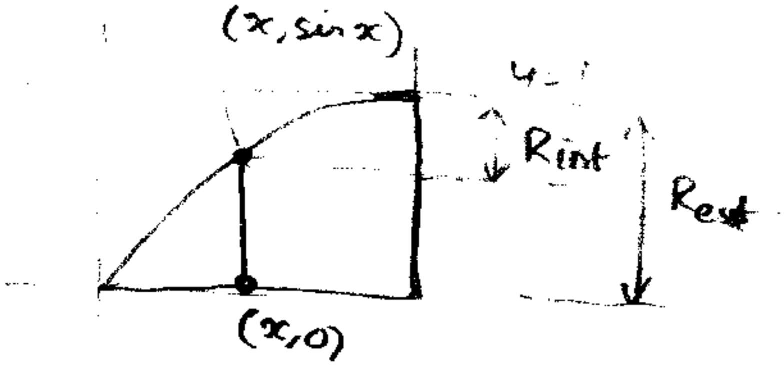
$$\frac{\pi^3}{4} - \pi\int_0^1 (\arcsin y)^2\,dy$$

$$= \frac{\pi^3}{4} - \pi \left[y \arcsin^2 y + 2 \arcsin y \sqrt{1-y^2} - 2y \right]_0^1$$

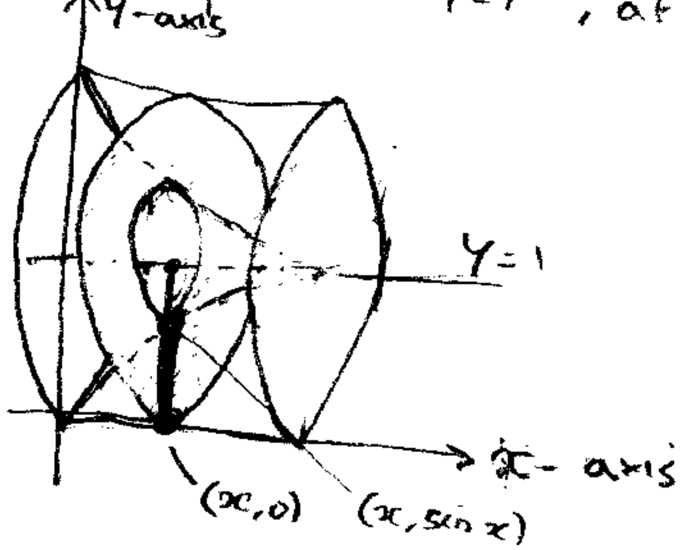
$$= \frac{\pi^3}{4} - \pi \left[\overbrace{\arcsin^2(1)}^{=\left(\frac{\pi}{2}\right)^2} + 2\arcsin(1) \overbrace{\sqrt{1-1}}^{=0} - 2 - \left(0 + \underbrace{2\arcsin(0)}_{=0} - 2 \times 0\right) \right]$$

$$= \frac{\pi^3}{4} - \frac{\pi^3}{4} + 2\pi = 2\pi$$

(iii) around the line $y=1$



Slice : perp. to $y=1$, at x :



annulus : $R_{int} = (1-\sin x)$

$R_{ext} = (1-0)$

→ area is $\pi(1^2 - (1-\sin x)^2)$

→ volume of solid of revolution is

$$\pi\int_0^{\pi/2} (1-(1-\sin x)^2)\,dx$$

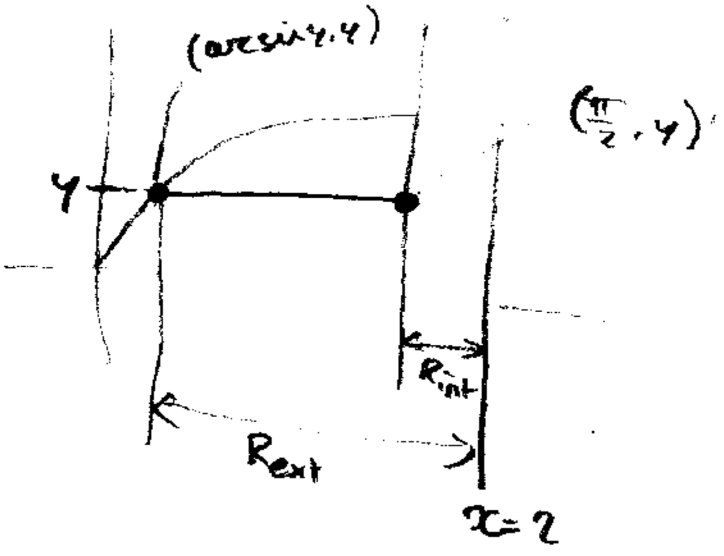
$$= \pi \int_0^{\pi/2} dx \quad - \pi \int_0^{\pi/2} 1 - 2\sin x + \sin^2 x \, dx$$

$$= \underbrace{\pi \left[x \right]_0^{\pi/2}}_{\substack{\shortparallel \\ \frac{\pi^2}{2}}} \; \underbrace{- \pi \int dx}_{\substack{\shortparallel \\ \frac{\pi^2}{2}}} \; + 2\pi \underbrace{\int_0^{\pi/2} \sin x \, dx}_{\substack{= \left[-\cos x \right]_0^{\pi/2} \\ = 1}} \; \underbrace{- \pi \int_0^{\pi/2} \sin^2 x \, dx}_{\substack{\text{From (1):} \\ \frac{\pi}{4}}}$$

$$= \frac{\pi^2}{2} - \frac{\pi^2}{2} + 2\pi - \frac{\pi^2}{4}$$

$$= 2\pi - \frac{\pi^2}{4}.$$

(iv) Rotated around $x=2$.



$R_{int} = 2 - \frac{\pi}{2}$

$R_{ext} = 2 - \arcsin y$.

as in (ii), we cut horizontal cross sections:

the annulus has area

$$\pi\left(R_{ext}^2 - R_{int}^2\right)$$

$$= \pi\left((2-\arcsin y)^2 - (2-\tfrac{\pi}{2})^2\right)$$

thus the volume of the solid of revolution is

$$\pi\int_0^1 (2-\arcsin y)^2 - (2-\tfrac{\pi}{2})^2 \, dy.$$

$$= \pi\int_0^1 (2-\arcsin y)^2 - \pi \underbrace{\int_0^1 (2-\tfrac{\pi}{2})^2 dy}_{=(2-\frac{\pi}{2})^2}$$

let's compute

$$\int_0^1 (2-\arcsin y)^2 dy$$

$$= \int_0^1 4 - 4\arcsin y + \arcsin^2 y \, dy$$

$$= 4 - 4\left[y\arcsin y + \sqrt{1-y^2}\right]_0^1 + \left[y\arcsin^2 y + 2\arcsin y\sqrt{1-y^2} - 2y\right]_0^1$$

See subquestion (ii)

$$= 4 - 4\left[\arcsin(1) + 0 - (0+1)\right] + \left[\left(\frac{\pi}{2}\right)^2 - 2\right]$$

$$= 4 - 4\left(\frac{\pi}{2} - 1\right) + \left(\frac{\pi}{2}\right)^2 - 2$$

$$= \frac{\pi^2}{4} - 2\pi + 6$$

Thus the volume we look for is

$$= \pi\left(\frac{\pi^2}{4} - 2\pi + 6 - \pi\left(2 - \frac{\pi}{2}\right)^2\right.$$

$$= \pi\left(\cancel{\frac{\pi^2}{4}} - \cancel{2\pi} + 6 - 4 + \cancel{2\pi} - \cancel{\frac{\pi^2}{4}}\right)$$

$$= 2\pi$$