How to know which convergence tests to apply for a series

Given $\sum a_n$, if you are asked to compute the value of the series if it converges, here are some hints:

- First thing first, the **limit** of a serie is the value

  $$L = \lim_{n\to\infty} \sum_{i=0}^{n} a_i.$$

  It is **not** the same as

  $$A = \lim_{n\to\infty} a_n.$$

  If the series converges, L is a number, and A is always null.

- Then there are mainly two types of series you can compute:

  1. The geometric series:

     $$\sum ax^n$$

  2. The series of the form

     $$\sum (d_n - d_{n-1})$$

  You may have to rearrange the terms to get back to one of those two cases, but that is it.

Now, sometimes you are only asked to **decide** whether

$$\sum a_n$$

converges or not. Then you have a handful of tests at your disposal. Here are some hints and comments:

- First if $\lim_{n\to\infty} a_n \neq 0$, the series diverge.
- If it contains only n^p's and radicals $\sqrt[p]{\cdots}$, you may want to try the limit comparison test.
- If it contains some factorials $n!$, the ratio test is a good guess.
- If you have only powers of n, e.g. $\frac{5^{3n+4}+2}{3^{5n}}$, try to get back to a geometric series.
- Any series of the form $\sum 1/n^p$ is a p-series: you know when this converges.
- If $a_n = f(n)$, with f a **decreasing** and **positive** function, the integral test might do the job.

- If b_n is decreasing to zero, $\sum(-1)^n b_n$ converges. It might not be absolutely convergent, though.
- **Be careful** when you have a sine or a cosine: $\lim_{n\to\infty}\cos(n)$ does not exist, thus the limit comparison test **might not work**! However, often the simple comparison test will. Make sure to choose wisely what should be below, depending on the convergence or the divergence of the serie:

$$0 \leq \sum \frac{1+\cos^2(n)}{n^2} \leq \sum \frac{2}{n^2}$$

versus

$$0 \leq \sum \frac{2}{n} \leq \sum \frac{3+\sin(n)}{n}$$