I was asked to solve an exercise from the list of suggested exercises on series, namely to decide of the convergence of

$$\sum \frac{n!}{n^n}.$$

Having a factorial, we try the ratio test, with $a_n = n!/n^n$:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{(n)!}{(n)^n}} = \frac{(n+1)!n^n}{n!(n+1)^{n+1}}.$$

Since $(n+1)^{n+1} = (n+1)(n+1)^n$ and since $(n+1)! = (n+1)(n!)$,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n!(n+1)n^n}{n!(n+1)^n(n+1)} = \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n.$$

Now, we want this to converge to something strictly smaller than 1, to be able to use the ratio test. We need some rewriting. Developping $(a+b)^n$ we get:

$$(1+b)^n = b^n + n \cdot b^{n-1} + \frac{n(n-1)}{2}b^{n-2} + \cdots$$

Replacing b by n we have

$$(n+1)^n = n^n + n \cdot n^{n-1} + \frac{n(n-1)}{2}n^{n-2} + \cdots$$

Since $n \cdot n^{n-1} = n^n$,

$$(n+1)^n = 2n^n + \frac{n(n-1)}{2}n^{n-2} + \cdots$$

Since

$$2n^n + \leq 2n^n + \frac{n(n-1)}{2}n^{n-2} + \cdots$$

we have

$$2n^n \leq (n+1)^n.$$

Hence,

$$\frac{n^n}{(n+1)^n} \leq \frac{n^n}{2n^n} = \frac{1}{2} < 1.$$

The ratio test can be apply:

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L \leq \frac{1}{2} < 1$$

thus the series $\sum n!/n^n$ is convergent.