Sums and Triangular Stacks of Integers

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1 Introduction

The game[1] is the following. I give you a triangular stack of bullets, upside-down, of some given height n, say $n = 3$, as follows.

$$\begin{array}{ccccc} \bullet & & \bullet & & \bullet \\ & \bullet & & \bullet & \\ & & \bullet & & \end{array}$$

Your goal is to replace the bullets with distinct integers from 1 to $n \cdot (n+1)/2$ such that the number below each pair of horizontal neighbors is the absolute value of their difference.

For $n = 2$, 3, 4 and 5, the following fillings are examples of solutions.

$$\begin{array}{ccc} 3 & & 1 \\ & 2 & \end{array} \qquad \begin{array}{ccccc} 2 & & 6 & & 5 \\ & 4 & & 1 & \\ & & 3 & & \end{array} \qquad \begin{array}{ccccccc} 9 & & 10 & & 3 & & 8 \\ & 1 & & 7 & & 5 & \\ & & 6 & & 2 & & \\ & & & 4 & & & \end{array} \qquad \begin{array}{ccccccccc} 6 & & 14 & & 15 & & 3 & & 13 \\ & 8 & & 1 & & 12 & & 10 & \\ & & 7 & & 11 & & 2 & & \\ & & & 4 & & 9 & & & \\ & & & & 5 & & & & \end{array}$$

Can you find more solutions?

For $n = 6$, the problem has no solution. Using a SMT solver[2], one can relatively easily show that there is no solution for $n = 6$, 7, 8 and 9 (timescale: half an hour). For n equal 10 and above, the solver is no longer efficient (it runs for more that 20 hours).

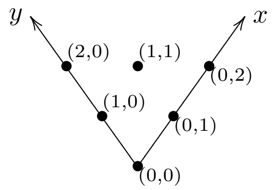
In the following, we prove (using regular math) that there is no solution for all $n \geq 6$.

[1] This game was first presented to me by Germain Faure as a problem used in some first-year computer science course for students to practice backtracking. He also implemented the problem on a SMT solver to check a few cases.

[2] SMT stands for "Satisfiability Modulo Theories". A SMT solver is a SAT solver where boolean variables are replaced with predicates in some classical first-order logic with equality.

2 Some formalism

We formalize the problem as follows: a triangle of height $n \geq 2$ is modeled by a subset $A_n = \{ (i, j) \mid i + j \leq n - 1 \}$ of $\mathbb{N} \times \mathbb{N}$. For $n = 3$, the triangle is embedded in the xy-plane as follows.



The exercise can be reformulated to finding a map $f : A_n \to \{ 1, \dots, \frac{n \cdot (n+1)}{2} \}$ such that

1. for all $(i, j) \in A_n$, the equality $f(i, j) = |f(i + 1, j) - f(i, j + 1)|$ holds;
2. the map f is bijective.

Condition (1) is equivalent to saying that for all $(i, j) \in A_n$,

$$f(i + 1, j) = f(i, j) + f(i, j + 1) \quad \text{or} \quad f(i, j + 1) = f(i, j) + f(i + 1, j).$$

We say that a function f satisfying Conditions (1) and (2) *defines a triangle of height n*.

Definition 2.1. Let (i, j) be a pair of integers. The *height* of (i, j) is the sum $i + j + 1$. If $f(i + 1, j) = f(i, j) + f(i, j + 1)$, we say that (i, j) is *left-oriented*, that $(i + 1, j)$ is the *target* of (i, j) and that $(i, j + 1)$ is *free* with respect to (i, j). If $f(i, j+1) = f(i, j)+f(i+1, j)$, we say that (i, j) is *right-oriented*, that $(i, j + 1)$ is the *target* of (i, j) and that $(i + 1, j)$ is *free* with respect to (i, j).

3 Chains and maximal chains

Suppose that $n \geq 3$ and that f defines a triangle of height n.

Definition 3.1. Let (i, j) be a pair in A_n. A (i, j)-*chain* is a list of pairs $(i_k, j_k)_{k=1}^{s}$ such that $i_1 = i$, $j_1 = j$, such that for all $k < s$, (i_{k+1}, j_{k+1}) is the target of (i_k, j_k) and such that $i_s + j_s = n - 1$. The *length* of the chain is s. The pair (i, j) is called the *origin* of the chain and (i_s, j_s) its *target*. The set of *free elements* of the chain is the set of the pairs free with respect to each (i_k, j_k) together with the origin (i, j). The set of elements of a chain union the set of their free elements is called the *trace of the chain*. The *maximal chain* is the chain starting at $(0, 0)$. The free elements of the maximal chain are called *small* and any non-small element is called *large*.

Convention 3.2. When this does not cause any confusion, we identify the elements of A_n with their images by f.

Example 3.3. In the following triangle, the maximal trace is marked in bold.

$$\begin{array}{ccccccccc} 6 & & 14 & & \mathbf{15} & & \mathbf{3} & & 13 \\ & 8 & & \mathbf{1} & & \mathbf{12} & & 10 & \\ & & 7 & & \mathbf{11} & & \mathbf{2} & & \\ & & & \mathbf{4} & & \mathbf{9} & & & \\ & & & & \mathbf{5} & & & & \end{array}$$

The target of the chain is 15, and the free elements are $1, 2, 3, 4$ and 5.

Lemma 3.4. *Let $(i_k, j_k)_{k=1}^{s}$ be any chain. The number of free elements of the chain is equal to s, which is also equal to $n - i_1 - j_1$.*

Proof. For all $k < s$, there is one element free with respect to each (i_k, j_k): either $(i_k + 1, j_k)$ or $(i_k, j_k + 1)$. Moreover, (i_1, j_1) is free. Adding all together, we get s free elements.

The fact that s is equal to $n - i_1 - j_1$ is proved by induction on s. The base case uses using the fact that $i_s + j_s = n - 1$. □

Lemma 3.5. *Let the maximal chain be $(i_k, j_k)_{k=1}^{s}$.*

1. *its length s is precisely n;*
2. *$f(i_n, j_n) = \frac{n\cdot(n+1)}{2}$;*
3. *for all k, the height of (i_k, j_k) is equal to k;*
4. *the image of the set of free elements of the chain is exactly $\{ 1, \ldots, n \}$.*

Proof. (1) Its length is precisely the height of the triangle. (2) The sum of n distinct integers greater or equal to one is at least equal to the sum $\sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$, that is, the maximum allowed for images by f. (3) By induction on the height of (i_k, j_k), using Definition 2.1. (4) If one of them is not in the set $\{ 1 \ldots n \}$, the total sum will exceed the maximum allowed. □

Corollary 3.6. *The value by f of any large element is at least $n + 1$.*

Proof. A large element is not small, and its image has to be distinct from the image of any other element, hence of any small element. Being larger or equal to 1, it must be larger than n. □

Lemma 3.7. *Let (i, j) be an element in A_n.*

1. *Any element (a, b) in the trace of a chain starting from (i, j) is such that $a \geq i$ and $b \geq j$.*
2. *If (a, b) is a element of a trace of a chain whose penultimate element is (i, j), then $a \leq i + 1$ and $b \leq j + 1$.*

Proof. Both items are proved by induction on the distance between (a, b) and (i, j), using the fact that the target of any element in A_n is a pair with either a strictly greater left component or a strictly greater right component. □

Lemma 3.8. *Any pair (a, b) in the trace of the maximal chain $(i_k, j_k)_{k=1}^n$ is such that $a \leq 1 + i_{n-1}$ and $b \leq 1 + j_{n-1}$.*

Proof. Corollary of Lemma 3.7. □

Corollary 3.9. *Let $(i_k, j_k)_{i=1}^n$ be the maximal chain and let (a, b) be such that $a \geq 2 + i_{n-1}$ or $b \geq 2 + j_{n-1}$. Then $f(a, b) \geq n + 1$.*

Proof. From Lemma 3.8 and Corollary 3.6, using the fact that any (a, b) not on the maximal trace is large. □

Lemma 3.10. *Consider a chain of length s, whose trace only contains large elements. Then the image of the target of the chain is at least equal to $\frac{1}{2} \cdot s \cdot (2 \cdot n + 1 + s)$.*

Proof. The image of the target is equal to the sum of the image of the free elements of the chain. This sum is greater or equal to $n+1+n+2+\ldots+n+s$, that is, $s \cdot n + \sum_{k=1}^{s} k$, which is equal to the required quantity. □

4 An upper bound for the existence of solutions

Definition 4.1. Let L_n^f be the set of pairs (a, b) such that $a \geq 2 + i_{n-1}$ and R_n^f the set of pairs (a, b) such that $b \geq 2 + j_{n-1}$.

Lemma 4.2. *The sets L_n^f and R_n^f only contain large elements.*

Proof. Direct consequence of Definition 4.2 and Corollary 3.9. □

Lemma 4.3. *The target of the maximal chain is neither contained in L_n^f nor in R_n^f.*

Proof. By Lemma 3.7, the target (i_n, j_n) of the maximal chain is such that $i_n \leq i_{n-1} + 1$ and $j_n \leq j_{n-1} + 1$. One can conclude with Definition 4.1. □

Lemma 4.4. *There exists a chain of at least $[(n-1)/2]$ free elements whose trace is contained in L_n^f or in R_n^f.*

Proof. Let $(i_k, j_k)_{i=1}^n$ be the maximal chain and suppose that $i_{n-1} \leq j_{n-1}$. Since $i_{n-1} + j_{n-1} + 1 = n - 1$ from the definition of the height in Definition 2.1 and Lemma 3.5, we can deduce that $i_{n-1} + 2 \leq n - j_{n-1}$. Since $n \geq 3$, from Lemma 3.5 we have $j_{n-1} > 1$. This makes $(i_{n-1} + 2, 0)$ in A_n.

By Lemma 3.7 the trace of the chain starting at $(i_{n-1} + 2, 0)$ is contained in L_n. From Lemma 3.4, the chain has $n - i_{n-1} - 2$ free elements. Since $i_{n-1} + j_{n-1} + 1 = n - 1$, the number of free elements is equal to j_{n-1}. Since $i_{n-1} \leq j_{n-1}$, it is larger or equal to $[(n-1)/2]$.

Now, suppose that $i_{n-1} \geq j_{n-1}$. The argument is symmetric, exchanging the roles of i_{n-1} and j_{n-1}, and the roles of L_n and R_n: the trace of the chain starting at $(0, j_{n-1})$ has at least $[(n-1)/2]$ free elements. $\square$

Using Lemmas 3.10 and 4.4, one can derive an upper-bound for the heights of valid triangles.

Lemma 4.5. *Let f define a triangle of height n, and consider a chain whose trace lies in L_n^f or in R_n^f. If $n = 9$, $n = 11$ or $n \geq 13$, its length has to be strictly smaller to $[\frac{n-1}{2}]$.*

Proof. By Lemma 4.3, the target of the maximal chain is not the target of the considered chain. Lemma 4.2 says that the free elements of this chain are large. Let P_n be the sequence of polynomials $P_n(s) = \frac{1}{2} \cdot s \cdot (2 \cdot n + 1 + s)$. From Lemma 3.10 it is enough to verify that for $n = 9$, $n = 11$ or $n \geq 13$, the value $P_n([\frac{n-1}{2}])$ is greater or equal to $\frac{n \cdot (n+1)}{2}$. They are two cases based on the parity of n.

Case $n = 2 \cdot k$. The inequality is

$$P_{2 \cdot k}(k-1) = \frac{5}{2} \cdot (k-1) \cdot k \geq k \cdot (2 \cdot k + 1)$$

and simplifies into

$$k \cdot (k-7) \geq 0,$$

with positive solutions $k \geq 7$. This means that for n even, if $n \geq 14$ then $P_n([\frac{n-1}{2}])$ is greater or equal to $\frac{n \cdot (n+1)}{2}$.

Case $n = 2 \cdot k + 1$. The inequality is

$$P_{2 \cdot k+1}(k) = \frac{1}{2} \cdot k \cdot (5 \cdot k + 3) \geq (2 \cdot k + 1) \cdot (k+1)$$

and simplifies into

$$k^2 - 3 \cdot k - 1 \geq 0,$$

with positive integer solutions $k \geq 4$. For n odd, this means that $P_n([\frac{n-1}{2}])$ is greater or equal to $\frac{n \cdot (n+1)}{2}$ when $n \geq 9$.

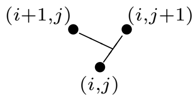
Finally, collecting the value for n with sufficient condition for $P_n([\frac{n-1}{2}])$ to be greater or equal than $\frac{n \cdot (n+1)}{2}$, we get the requested values. $\square$

Corollary 4.6. *There does not exist any valid triangle for heights $n = 9$, $n = 11$ and $n \geq 13$.*

Proof. By Lemma 4.4, there always exists a chain whose trace lies in L_n^f or in R_n^f and whose length is at least $[\frac{n-1}{2}]$. Lemma 4.5 states that for valid triangles of heights $n = 9$, $n = 11$ and $n \geq 13$, such a chain cannot exists. Therefore, no valid triangle exists for these heights. $\square$

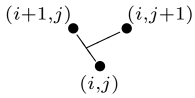
5 More notations and results

From here, we are almost done. It remains to show that the cases $n = 6, 7, 8, 10$ and 12 are unsatisfiable. We will use the diagrams



to say that (i, j) is left-oriented,

and



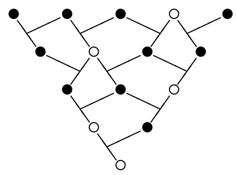
to say that (i, j) is right-oriented.

Bullets $\bullet$ will also be used to symbolize large elements and circles $\circ$ small elements.

Example 5.1. The triangle

6 14 15 3 13
8 1 12 10
7 11 2
4 9
5

satisfies the set of constraints



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Lemma 5.2. *Consider a triangle of height $n \geq 1$ and a chain whose trace only contains large elements. Then its length is strictly smaller to to $[\frac{n-1}{2}] + 1$.*

Proof. Proof done as for Lemma 4.5. □

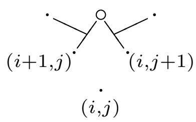
Lemma 5.3. *Consider a triangle of height $n \geq 1$. For each element (i, j) of the maximal chain not equal to $(0, 0)$, either $(i + 1, j - 1)$ or $(i - 1, j + 1)$ is small. Together with the small element $(0, 0)$, this exhausts all the small elements of the triangle.*

Proof. By definition of small elements. □

Lemma 5.4. *Consider a triangle of height $n \geq 1$. Then there is only one small element for a given height.*

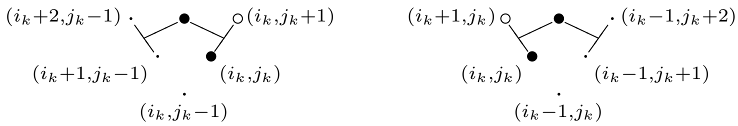
Proof. By definition of the maximal chain, there is only one element of the maximal chain at a given height. From Lemma 5.3, this implies that there is only one small element for a given height. □

Lemma 5.5. *Consider a triangle of height $n \geq 3$ and let $(i + 1, j + 1)$ be a small element. Then $(i, j + 1)$ is right-oriented and $(i + 1, j)$ is left-oriented, as summarized in the following diagram.*



Proof. From Lemma 5.4 the points $(i+2, j)$ and $(i, j+2)$ are both large. Thus either the sum of the images of $(i+2, j)$ and $(i+1, j)$ or the sum of the images of $(i, j+2)$ and $(i, j+1)$ is strictly greater than n. Therefore, $(i, j+1)$ and $(i+1, j)$ cannot be respectively left and right oriented: they are respectively right and left oriented. □

Lemma 5.6. *Consider a triangle of height $n \geq 3$ and let $(i_k, j_k)_{k=1}^n$ be the maximal chain. Consider $k \geq 3$ such that $i_k, j_k \geq 1$. If (i_k, j_k) is left-oriented (respectively right-oriented), then (i_k+1, j_k-1) is right-oriented (respectively (i_k-1, j_k+1) is left-oriented). This is summarized by the following diagrams:*



Proof. We prove the lemma in the case where (i_k, j_k) is left-oriented: it is similar in the right-oriented case by exchanging the role of i and j. Suppose that the element (i_k+1, j_k-1) is left-oriented. it generates a chain whose free elements contains (i_k+1, j_k) and whose target is the sum of $n-(i_k+1+j_k-1) = n-i_k-j_k$ elements. The element (i_k+1, j_k) belongs to the maximal chain, and its image is the sum of i_k+j_k+2 (small) elements. Thus the target of the chain generated by (i_k+1, j_k-1) is the sum of $n+1$ elements. All these elements being distinct, the sum must be greater than $\frac{n\cdot(n+1)}{2}$: this is not possible. Therefore, since (i_k+1, j_k-1) cannot be left-oriented, it is right-oriented. □

6 Valid triangles of height $n = 6$

In this section we give a set of constraints that a valid triangle of height $n = 6$ must satisfy and show that this set is inconsistent. We build the set of constraints step by step as in Figures 1 and 2. The conventions are as follows: dots $\cdot$ are unknown values, bullets $\bullet$ are elements of the maximal chain, circles $\circ$ are small elements and stars $\star$ are other large values.

Step 1 The triangle is empty.

Step 2 Without loss of generality, the penultimate element of the maximal chain is centered. Otherwise, one can find a chain of large elements of length 3. Using Lemma 5.2, one reaches a contradiction. Again without loss of generality, one can choose this penultimate element to be left-oriented (the argument is symmetric by exchanging the role of the first and the second coordinate).

Step 3 The fourth element of the maximal chain is either $(1, 2)$ or $(2, 1)$. In the first case, one can again exhibit a chain of large elements with length 3, reaching a contradiction. The fourth element is then $(2, 1)$.

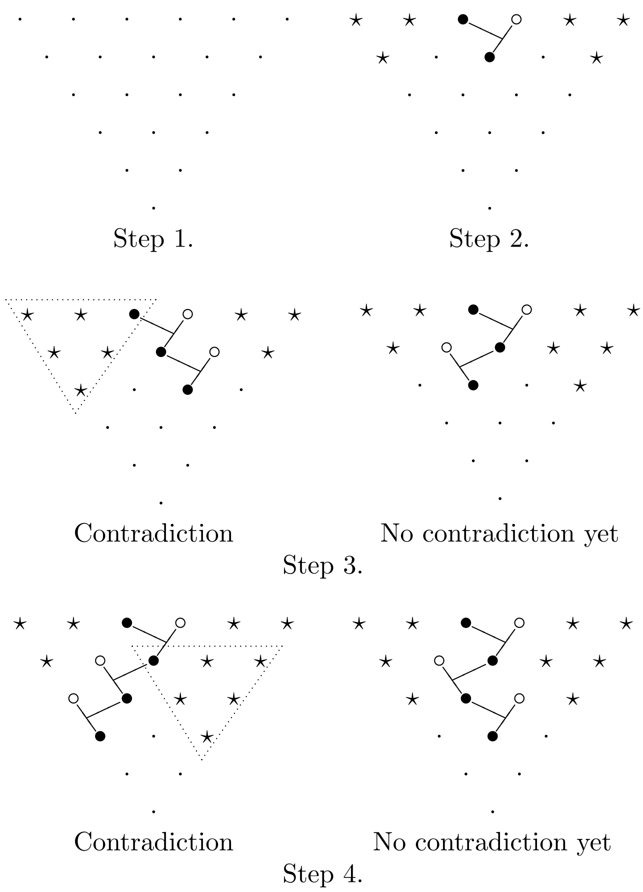


Figure 1: Building the set of constraints for a valid triangle of height $n = 6$, Steps 1 to 4.

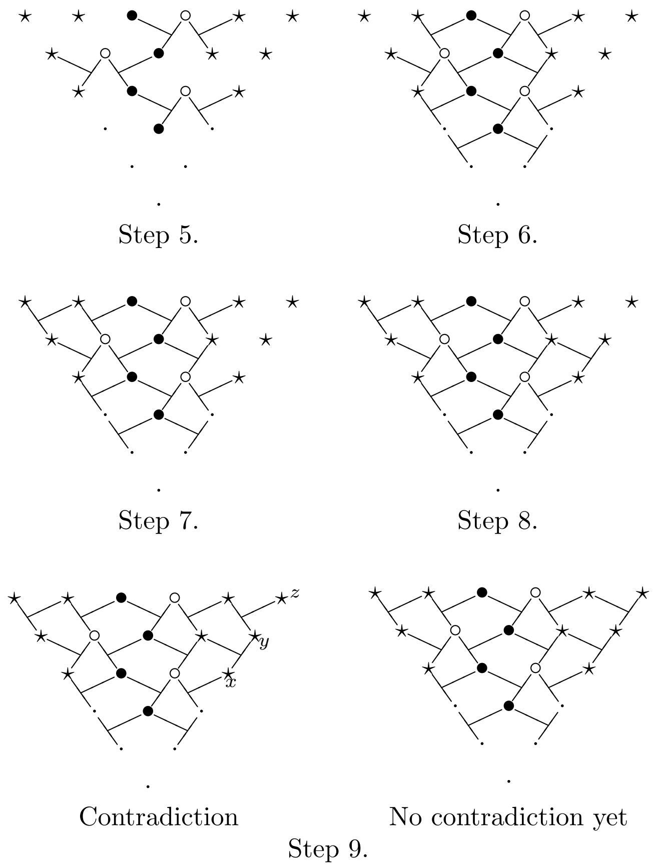


Figure 2: Building the set of constraints for a valid triangle of height $n = 6$, Steps 5 to 9.

Step 4 We look for the third element of the chain. It cannot be $(2, 0)$ since one could then produce a large chain of length 3; it must be $(1, 1)$.

Step 5 One applies Lemma 5.5.

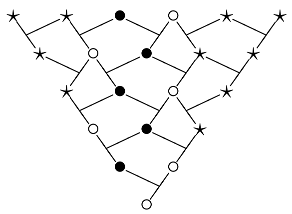
Step 6 One applies Lemma 5.6.

Step 7 If $(5, 0)$ were left-oriented, the image of $(6, 0)$ would be greater than 21.

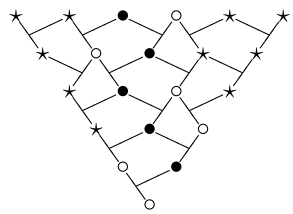
Step 8 If $(0, 3)$ were right-oriented, it would generate a large chain of length 3.

Step 9 The element $(4, 0)$ cannot be right-oriented. Otherwise, the value z is greater or equal to $x + 2 \cdot y$, where x and y are large elements, thus greater or equal to 7. Since all values must be at most equal to 21, the value z must then be equal to 21, which is not possible since there is already an element of value 21. The element $(4, 0)$ is therefore left-oriented.

We are left with the two possible set of constraints



and



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If we assign to the elements the values $x_1, \ldots, x_{21}$, left-to-right, top-to-bottom as follows

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \\ x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \\ x_{12} \quad x_{13} \quad x_{14} \quad x_{15} \\ x_{16} \quad x_{17} \quad x_{18} \\ x_{19} \quad x_{20} \\ x_{21} \end{array},$$

then the two set of constraints can be rewritten as respectively Table 1 and Table 2.

First set The constraints in Table 1 can be reformulated in terms of the small elements $x_4, x_8, x_{14}, x_{16}, x_{20}, x_{21}$. From this system, the equation

$$x_1 + x_{11} = x_4 + x_{16} + x_{20} - x_{14}$$

can be inferred. The left-hand-side consists only of large elements. The sum is then at least equal to $7 + 8 = 15$. The right-hand-side consists of small elements. It is thus at most equal to $6 + 5 + 4 - 1 = 14$. This makes the equation unsatisfiable: The first set of constraints does not have any solution.

$$
\begin{aligned}
x_1 &= x_4 + x_{16} - x_8 \\
x_2 &= x_4 + x_{14} + x_{16} + x_{20} + x_{21} \\
x_3 &= x_4 + x_8 + x_{14} + x_{16} + x_{20} + x_{21} \\
x_5 &= x_4 + x_8 + x_{16} + x_{20} + x_{21} \\
x_6 &= x_4 + x_{14} + x_{16} + x_{21} \\
x_7 &= x_8 + x_{14} + x_{20} + x_{21} \\
x_9 &= x_8 + x_{14} + x_{16} + x_{20} + x_{21} \\
x_{10} &= x_8 + x_{16} + x_{20} + x_{21} \\
x_{11} &= x_8 + x_{20} - x_{14} \\
x_{12} &= x_{14} + x_{20} + x_{21} \\
x_{13} &= x_{14} + x_{15} + x_{20} + x_{21} \\
x_{15} &= x_{14} + x_{16} + x_{21} \\
x_{17} &= x_{16} + x_{20} + x_{21} \\
x_{18} &= x_{16} + x_{21} \\
x_{19} &= x_{20} + x_{21}
\end{aligned}
$$

Table 1: First possible set of constraints for a valid triangle of height $n = 6$

$$
\begin{aligned}
x_1 &= x_4 + x_{18} + x_{21} - x_8 \\
x_2 &= x_4 + x_{14} + x_{18} + x_{19} + x_{21} \\
x_3 &= x_4 + x_8 + x_{14} + x_{18} + x_{19} + x_{21} \\
x_5 &= x_4 + x_8 + x_{18} + x_{19} + x_{21} \\
x_6 &= x_4 + x_{14} + x_{18} \\
x_7 &= x_8 + x_{14} + x_{19} \\
x_9 &= x_8 + x_{14} + x_{18} + x_{19} + x_{21} \\
x_{10} &= x_8 + x_{18} + x_{19} + x_{21} \\
x_{11} &= x_8 + x_{19} + x_{21} - x_{14} \\
x_{12} &= x_{14} + x_{19} \\
x_{13} &= x_{13} + x_{18} + x_{19} + x_{21} \\
x_{15} &= x_{14} + x_{18} \\
x_{16} &= x_{18} + x_{21} \\
x_{17} &= x_{18} + x_{19} + x_{21} \\
x_{20} &= x_{19} + x_{21}
\end{aligned}
$$

Table 2: Second possible set of constraints for a valid triangle of height $n = 6$

Second set The constraints in Table 2 can be reformulated in terms of the small elements $x_4, x_8, x_{14}, x_{18}, x_{19}, x_{21}$. Therefore the equation

$$x_1 + x_6 + x_7 + x_{11} + x_{12} + x_{15} + x_{16} + x_{20}$$
$$= 2 \cdot x_4 + x_8 + 3 \cdot x_{14} + 4 \cdot x_{18} + 3 \cdot x_{19} + 4 \cdot x_{21}$$

must be satisfied. The left-hand-side consists only of large elements. The sum is then at least equal to $7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 = 84$. The right-hand-side consists of small elements. It is thus at most equal to $4 \cdot 6 + 4 \cdot 5 + 3 \cdot 4 + 3 \cdot 3 + 2 \cdot 2 + 1 = 79$. This makes the equation unsatisfiable: The second set of constraints does not have any solution.

This concludes the proof that there does not exist any valid triangle of height $n = 6$: the set of constraints is inconsistent.

7 Valid triangles of height $n = 7$

A valid triangle of height $n = 7$ contains 28 elements and satisfies the following property.

Lemma 7.1. *Given any chain of length* 4 *whose set of free elements contains only one small element, the largest element of the chain can only be equal to* 28.

Proof. The highest element of a chain of length 4 is the sum of 4 free elements. If 3 of them are large and one is small, the sum of them is at least $8+9+10+1 = 28$. The largest possible element being 28, this is the value of the top element of the chain. □

We proceed as in the case $n = 6$ to get a list of constraints for a valid triangle of height $n = 7$ and shows that these constraints make an inconsistent set. We follow the same conventions as for the case $n = 6$.

Step 1 The triangle is empty.

Step 2 Without loss of generality, the last element of the maximal chain is centered. Otherwise, one can find a chain of large elements of length 4, and one reaches a contradiction using Lemma 5.2. Again without loss of generality, one can choose the penultimate element to be left-oriented (the argument is symmetric by exchanging the role of the first and the second coordinate). This fixes the seventh and sixth elements.

Step 3 The fifth element of the chain is either $(3, 1)$ or $(2, 2)$. Using Lemma 5.2, we can rule out the element $(3, 1)$: one could exhibit a sub-triangle of height 4.

Step 4 The fourth element of the chain is either $(2, 1)$ or $(1, 2)$.

Step 5 One applies Lemmas 5.5 and 5.6 on both cases and Lemma 7.1 on the element $(3, 0)$ of Case b.

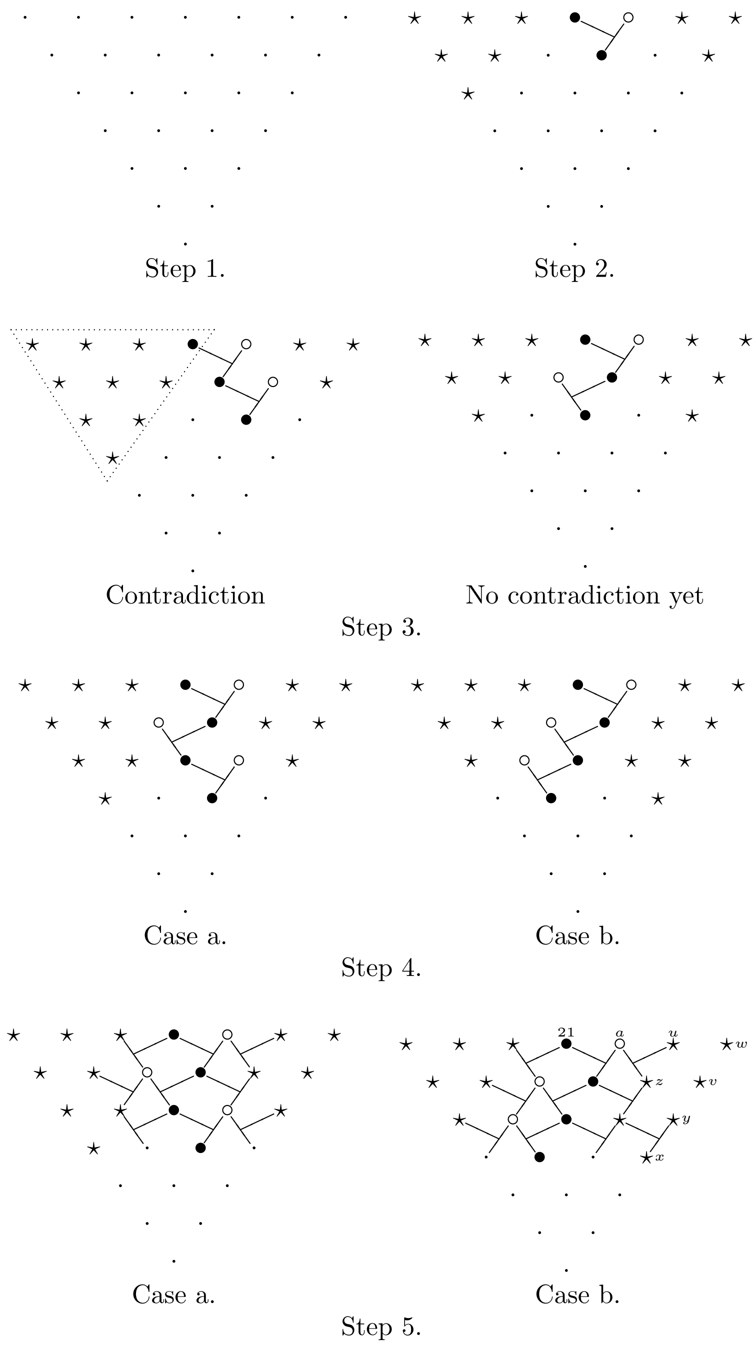


Figure 3: Set of constraints for a valid triangle of height $n = 7$, Steps 1 to 5.

Now, one can rule out both cases, as follows.

Case a If the element $(0, 3)$ were left-oriented then by Lemma 3.7 all the free elements of the generated chain would be large which is impossible from Lemma 3.10. If the element $(0, 3)$ were right-oriented, the generated chain would satisfy the hypothesis of Lemma 7.1. Its maximal element would be 28: by Lemma 3.7 this is not possible.

Case b Since a is small, x, y, z are large, and $a+x+y+z = 28$, one must have $a = 1$ and x, y, z ranging over $8, 9, 10$. Since $u = a + z = 1 + z$ and since u must be different from any other value, one must have $z = 10$ and $u = 11$. Now, since v is large and y is at least equal to 8, y must be left-oriented so that $v = z + y = 10 + y$. Similarly, v is left-oriented and $w = u + v$. But then $w = u + z + y = 11 + 10 + y$, where $y = 8$ or $y = 9$: this means that $w \geq 29$, which is impossible.

This concludes the proof that there does not exist any valid triangle of height $n = 7$.

8 Valid triangles of height $n = 8$

A valid triangle of height $n = 8$ contains 36 elements. We shall exhibit a set of constraints verified by an hypothetical valid triangle of height $n = 8$ and then show that this set is inconsistent. We use the same notation as in the two previous cases.

The same analysis as in case $n = 6$ and $n = 7$ applies and is summarized in Figure 4. The penultimate element of the maximal chain is determined using Lemma 5.2. Without loss of generality one can choose it to be left-oriented, and using Lemma 5.2 one can give the constraints in Step 1. Step 2 is obtained applying Lemmas 5.5 and 5.6 and Step 3 using Lemma 3.10.

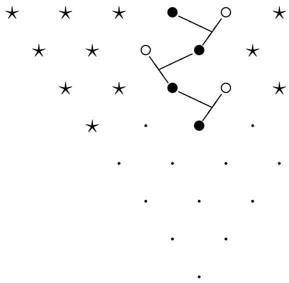
Now, we have $36 = b + a + x + y + z$, therefore $t = 36 - b$. Since $x > b$ the element x cannot be right-oriented: it would make $u = t + x > 36$. Therefore we have $t = x + u$ and thus $u = 36 - b - x$.

Since $36 = b+a+x+y+z$ with x, y, z greater or equal to 9 and a, b greater or equal to 1, $a + b \geq 3$ and $x + y + z \leq 33$. This means that x, y, z can take the values 9 to 14. This gives a maximum value of 5 to $|x - y|$. Being large, v cannot take it and the element y is thus right-oriented, and $v = x + y$.

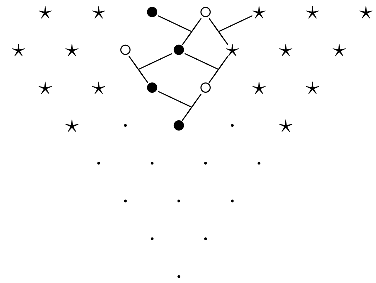
Finally, $u + v = 36 - b - x + x + y = 36 - b + y$. Since $y > b$, this cannot be the value w. Therefore $w + v = u$.

We can rewrite the value t as follows. $t = x + u = x + (w + v) = x + (w + (x + y)) = 2 \cdot x + y + w$. Since x, y and w are large, t is at least equal to $2 \cdot 9 + 10 + 11 = 39$. This is impossible.

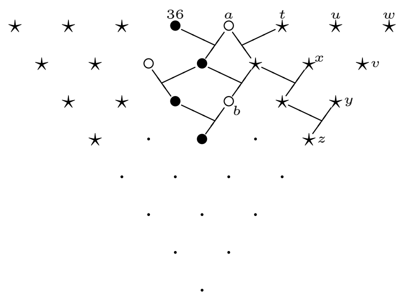
The set of constraints for a triangle of height $n = 8$ being unsatisfiable, there does not exist any triangle of such height.



Step 1.



Step 2.

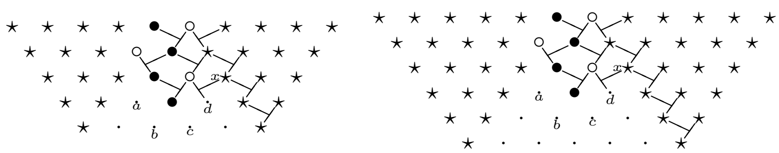


Step 3.

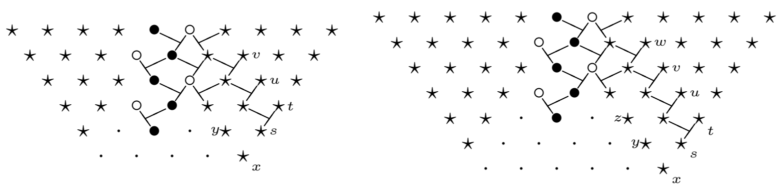
Figure 4: Set of constraints for a valid triangle of height $n = 8$, Steps 1 to 3.

9 Valid triangles of height $n = 10$ and $n = 12$

The constraints for triangles of height $n = 10$ and $n = 12$ can be deduced similarly to reach the following sets (the bottoms of the triangles are not drawn).



Either b or c is the previous element in the maximal chain, in which case a or d are respectively small. But d cannot be small (or x becomes the sum of two large elements and two small elements).



If the element x were right-oriented, it would generate a chain with one small element and more large elements than possible: 5 large elements in the case $n = 10$ and 6 in the case $n = 12$. This means that x is left-oriented. The same argument can be used for showing that the element y (and the element z in the case $n = 12$) must be left-oriented.

The element n is then greater or equal to the sum of x, s, t, u, v in the case $n = 10$ and x, s, t, u, v, w in the case $n = 12$. Since they are all large, n is greater than the allowed maximum.

The sets of constraints for valid triangles of height $n = 10$ and $n = 12$ are therefore inconsistent: there does not exists valid triangles for such heights.

10 Generalization

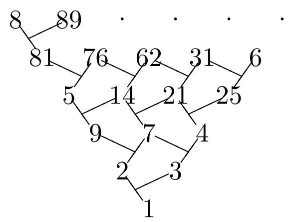
We present two generalizations of the constraints given in Section 2.

10.1 Infinite triangle

Here we allow the height n to be infinite: the set A_∞ is just $\mathbb{N} \times \mathbb{N}$, and the codomain of the function f is $\mathbb{N} \setminus \{ 0 \}$ (The function f is then a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N} \setminus \{ 0 \}$). It is possible to construct a valid infinite triangle by induction on the height of its rows of elements, as follows:

- The elements of each row are alternatively all left-oriented or all right-oriented.
- The first row is right-oriented.
- If a row is left-oriented, its right-most element is equal to the smallest value not yet used in the rows below it.
- If a row is right-oriented, its left-most element is equal to the smallest value not yet used in the rows below it.

The described triangle starts as follows:

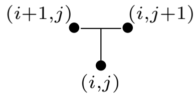


The triangle is filled row by row, alternatively from left to right and from right to left. At each step, either we write a value that is the sum of something and the maximum of all the values already written, or we write a value carefully chosen not to be already written. We are therefore sure not to write twice the same value.

We write all the positive natural numbers in the triangle. Indeed, suppose that some are not written. Consider the smallest one and call it x_0. At some stage in the filling of the triangle we will write a number larger than x_0. To start the next row, we will use the smallest of all the number not yet used, and it would be precisely x_0. We reach a contradiction: all the positive natural numbers will be written in the triangle.

10.2 Weak triangles

Here we generalize Condition (1) of Section 2: we also allow $f(i,j) = f(i+1,j) + f(i,j+1)$, in which case we say that (i,j) is down-oriented. Using the notation of Section 5, we can denote it as



A triangle verifying this weakened condition is called *weak*. A triangle without it is called *strict*.

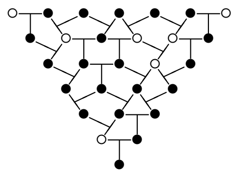
An exhaustive search for valid weak triangles finds answers for heights that were not possible in the strict case. Here is a summary.

Triangle height	2	3	4	5	6	7	8	9	10
Number of sol.	6	42	246	744	0	1158	2088	≥ 506	≥ 2

Note that that there is neither strict nor weak triangle of height 6.

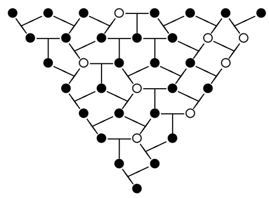
Examples of triangles for heights 7, 8, 9 and 10 are given in Figure 5, together with the corresponding set of constraints. For each height n, we use circles $\circ$ for elements up to n and bullets $\bullet$ for the larger ones.

5 18 21 13 19 17 7
23 3 8 6 2 24
20 11 14 4 26
9 25 10 22
16 15 12
1 27
28



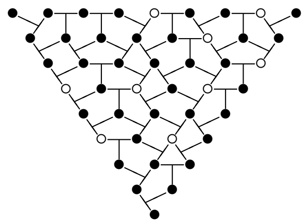
Height 7.

22 35 21 3 9 33 25 32
13 14 18 12 24 8 7
27 4 30 36 16 1
23 34 6 20 15
11 28 26 5
17 2 31
19 29
10



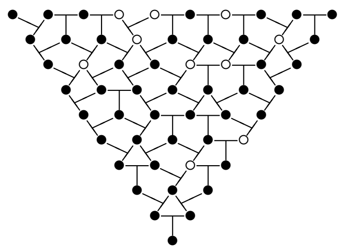
Height 8.

31 13 16 25 5 30 38 4 44
18 29 41 20 35 8 42 40
11 12 21 15 43 34 2
1 33 6 28 9 36
32 39 22 19 45
7 17 3 26
24 14 23
10 37
27



Height 9.

40 13 33 6 9 44 1 34 30 25
27 46 39 3 53 45 35 4 55
19 7 36 50 8 10 31 51
12 29 14 42 18 41 20
17 43 28 24 23 21
26 15 52 47 2
11 37 5 49
48 32 54
16 22
38



Height 10.

Figure 5: Examples of valid weak triangles