QDCS - Intro Quantum
Homework # 1
Due on Friday, Oct 10 2025, 9am

Recall that $\mathcal{H}$ is the Hilbert space generated from $|0\rangle, |1\rangle$.

1 Basis (3 pts)

Show that the following vectors form an orthonormal basis for $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ (known as the SHIFT basis)

$$|000\rangle, |111\rangle, |{+}01\rangle, |{-}01\rangle, |1{+}0\rangle, |1{-}0\rangle, |01{+}\rangle, |01{-}\rangle,$$

where for example $|01{+}\rangle$ stands for $|0\rangle \otimes |1\rangle \otimes |+\rangle$.

Do as few computations as possible.

2 Reversible Computation (7 pts)

Consider the set $\mathcal{B} \triangleq \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and the bijection σ on the set $\mathcal{B}$ defined as e_k to $e_{3k \bmod 8}$. In other word, it is defined as

$$\begin{array}{cccc} e_0 \mapsto e_0 & e_2 \mapsto e_6 & e_4 \mapsto e_4 & e_6 \mapsto e_2 \\ e_1 \mapsto e_3 & e_3 \mapsto e_1 & e_5 \mapsto e_7 & e_7 \mapsto e_5 \end{array}$$

Let $\mathcal{E}$ be the Hilbert space generated by $\mathcal{B}$, and U the linear map $\mathcal{E} \to \mathcal{E}$ defined as

$$U : |e_k\rangle \mapsto |\sigma(e_k)\rangle \qquad \text{for } k = 0, \ldots, 7$$

1. Write a matrix for U using the basis ordering $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$. (1 pt)
2. What does it means when $U_{m,n} = 1$? (1 pt)
3. Explain why U is unitary. (2 pts)
4. U can be regarded as an action on a register of 3 qubits. Design a circuit for U, acting on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, using only X, CNOT and Toffoli gates. Even if they are not necessary (in fact, 4 gates are enough), you can use ancillas but beware: we want a unitary circuit. As an encoding on bitstring, set the least significant bit on the right. (3 pts)

3 Measuring 2 Qubits (4 pts)

Consider the two-qubit system

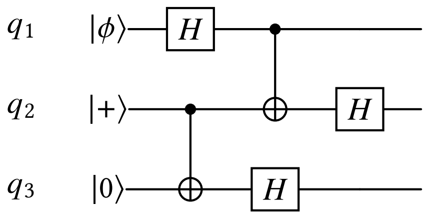
$$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle.$$

Show the detailed computation of why measuring the first qubit THEN the second qubit yields the same result as measuring the second qubit THEN the first qubit.

4 An example of measure

(3 pts)

Let $|\phi\rangle$ be $\alpha|0\rangle + \beta|1\rangle$, and consider the circuit

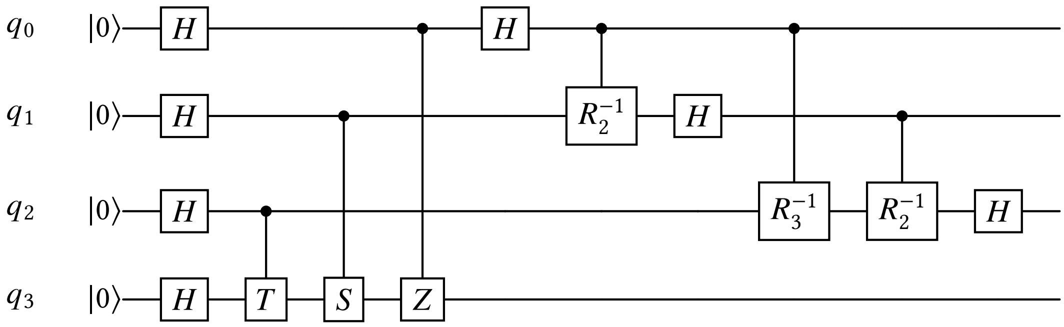


If we measure q_1 and q_2 at the end of the circuit, what is the result?
Show your computation.

5 A Mysterious Circuit

(3 pts)

Consider the following circuit.



where R_n is

$$R_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{2\pi}{2^n}} \end{pmatrix}.$$

Using what you know from QPE, give the state-vector resulting from running the circuit. Explain your reasoning.