MAT 1322 3X

Summer 2007

HMW #1.

SOLUTIONS

(1)(2) $\displaystyle\int \frac{x^3+x^2+4}{x^2+2x+5}\,dx$ ← partial Fractions.

First, need to do long division since $\deg(\text{num}) \geq \deg(\text{denom})$:

$$
\begin{array}{r}
x-1 \\
x^2+2x+5 \overline{\big) \; x^3+x^2+4} \\
-(x^3+2x^2+5x) \\
\hline
-x^2-5x+4 \\
-(-x^2-2x-5) \\
\hline
-3x+9
\end{array}
$$

Thus $\displaystyle\frac{x^3+x^2+4}{x^2+2x+5} = x-1 + \frac{9-3x}{x^2+2x+5}$

$\displaystyle = x-1-3\frac{x+3}{x^2+2x+5}$

$\displaystyle\frac{d}{dx}(x^2+2x+5) = 2x+2,$

thus we write $\displaystyle\frac{x+3}{x^2+2x+5} = \frac{1}{2}\,\frac{2x+6}{x^2+2x+5}$

$\displaystyle = \frac{1}{2}\,\frac{2x+2}{x^2+2x+5} + \frac{1}{2}\,\frac{4}{x^2+2x+5}.$

and then

$$\int \frac{x^3+x^2+4}{x^2+2x+5}\,dx = \underbrace{\int (x-1)\,dx}_{=\frac{x^2}{2}-x} \; \underbrace{-\frac{3}{2}\int \frac{2x+2}{x^2+2x+5}\,dx}_{-\frac{3}{2}\ln(x^2+2x+5)} \; \underbrace{-6\int \frac{dx}{x^2+2x+5}}_{?}$$

See next page.

To solve $\int \frac{dx}{x^2+2x+5}$, one proceeds with square completion:

$$x^2+2x+5 = x^2+2x+1+4$$
$$= (x+1)^2+2^2.$$

Thus $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{(x+1)^2+4}$

$x+1 = 2\tan u$ $\qquad u = \arctan(\frac{x+1}{2})$

$dx = 2\sec^2 du$

$$= \int \frac{2\sec^2 u \, du}{4(\tan^2 u+1)} = \int \frac{1}{2} du = \frac{1}{2}u + C$$
$$= \frac{1}{2}\arctan(\frac{x+1}{2}) + C.$$

Thus

$$\int \frac{x^3+x^2+4}{x^2+2x+5}dx = \frac{x^2}{2} - x - \frac{3}{2}\ln(x^2+2x+5)$$
$$- 3\arctan(\frac{x+1}{2}) + C$$

(ii) $\int \frac{x-2}{(2x^2+x+2)(x+1)^2}\,dx$ → we use partial Fractions

the denominator is already Factorized, and
$\deg(x-2) < \deg((2x^2+x+2)(x+1)^2)$

Thus one can find A, B, C and D such that

$$\frac{x-2}{(2x^2+x+2)(x+1)^2} = \frac{Ax+B}{2x^2+x+2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}$$

$$= \frac{(Ax+B)(x+1)^2 + C(2x^2+x+2)(x+1) + D(2x^2+x+2)}{(2x^2+x+2)(x+1)^2}$$

Knowing that $(x+1)^2 = x^2+2x+1$, the numerators becomes

$$\begin{aligned} x-2 = {} & x^3(A+2C) \\ & + x^2(2A+B+2C+C+2D) \\ & + x(A+2B+2C+C+D) \\ & + (B+2C+2D) \end{aligned}$$

and then A, B, C and D satisfy:

$$\begin{cases} 0 = A+2C & (1) \\ 0 = 2A+B+3C+2D & (2) \\ 1 = A+2B+3C+D & (3) \\ -2 = B+2C+2D & (4) \end{cases}$$

(1) → $A = -2C$: thus (2), (3), (4) becomes

$$\begin{cases} 0 = B-C+2D & (5) \\ 1 = 2B+C+D & (6) \\ -2 = B+2C+2D. & (7) \end{cases}$$

From (5), $C = B + 2D$.

Thus (7) is $-2 = (B+2D) + 2C = C + 2C = 3C$

and $\boxed{C = -\frac{2}{3}}$

Then $\boxed{A = \frac{4}{3}}$

(5) and (6) becomes, knowing $C = -2/3$:

$$\begin{cases} 0 = B + \frac{2}{3} + 2D \\ 1 = 2B - \frac{2}{3} + D \end{cases} \iff \begin{cases} B = -\frac{2}{3} - 2D & (8) \\ \frac{5}{3} = 2B + D. & (9) \end{cases}$$

(9) becomes, since $B = -\frac{2}{3} - 2D$:

$$\frac{5}{3} = -\frac{4}{3} - 4D + D$$

$$\iff \frac{9}{3} = -3D$$

$$\iff \boxed{D = -1}$$

If $D = -1$, then $B = -\frac{2}{3} + 2$: $\boxed{B = \frac{4}{3}}$

Hence $\dfrac{x-2}{2x^2+x+2} = \dfrac{4x+4}{3(2x^2+x+2)} - \dfrac{2}{3}\dfrac{1}{x+1} - \dfrac{1}{(x+1)^2}$.

Now, since $\frac{d}{dx}(2x^2+x+2) = 4x+1$,

we split $\dfrac{4x+4}{3(2x^2+x+2)} = \dfrac{4x+1}{3(2x^2+x+2)} + \dfrac{1}{2x^2+x+2}$.

and $\displaystyle\int \frac{x-2}{(2x^2+x+2)(x+1)^2}\,dx$

$$= \underbrace{\frac{1}{3}\int \frac{4x+1}{2x^2+x+2}\,dx}_{=\frac{1}{3}\ln(2x^2+x+2)} + \int \frac{dx}{2x^2+x+2} - \underbrace{\frac{2}{3}\int \frac{dx}{x+1}}_{-\frac{2}{3}\ln|x+1|} - \underbrace{\int \frac{dx}{(x+1)^2}}_{+\frac{1}{x+1}} + C$$

we proceed to square completion:

$$2x^2+x+2 = 2\left(x^2+\frac{1}{2}x+1\right)$$
$$= 2\left(x^2+\frac{1}{2}x+\frac{1}{16}+\frac{15}{16}\right)$$
$$= 2\left(\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2\right)$$

Thus $\displaystyle\int \frac{dx}{2x^2+x+2} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{15}{16}}$

$$x+\frac{1}{4} = \frac{\sqrt{15}}{4}\tan u \quad \rightarrow u = \arctan\left(\frac{4x+1}{\sqrt{15}}\right)$$
$$\rightarrow dx = \frac{\sqrt{15}}{4}\sec^2 u\, du$$

$$= \frac{1}{2}\int \frac{\frac{\sqrt{15}}{4}\sec^2 u\, du}{\frac{15}{16}\tan^2 u+\frac{15}{16}} = \frac{1}{2}\times\frac{\sqrt{15}}{4}\times\frac{16}{15}\int 1\cdot du$$
$$= \frac{2}{\sqrt{15}}u + C$$
$$= \frac{2}{\sqrt{15}}\arctan\left(\frac{4x+1}{\sqrt{15}}\right).$$

Thus $\displaystyle\int \frac{x-2}{(2x^2+x+2)(x+1)^2}\,dx = \frac{1}{3}\ln(2x^2+x+2)+\frac{2}{\sqrt{15}}\arctan\left(\frac{4x+1}{\sqrt{15}}\right)-\frac{2}{3}\ln|x+1|+\frac{1}{x+1}+C$

(iii) $\int \frac{x^3}{(x+7)^3}\,dx$

$\deg(x^3) = \deg(x+7)^3$; we proceed with long division.

$$(x+7)^3 = x^3 + 3\times 7\times x^2 + 3\times 7^2\times x + 7^3$$
$$= x^3 + 21x^2 + 147x + 343$$

$$x^3+21x^2+147x+343 \;\overline{)\; x^3}\quad 1$$
$$-(x^3+21x^2+147x+343)$$
$$-21x^2-147x-343.$$

Thus $\frac{x^3}{(x+7)^3} = 1 - \frac{3\times 7x^2 + 3\times 7^2 x + 7^3}{(x+7)^3}.$

$$= 1 - \frac{(x+7)^3 - x^3}{(x+7)^3}$$

Now there exists A, B and C such that

$$\frac{3\times 7x^2 + 3\times 7^2 x + 7^3}{(x+7)^3} = \frac{A}{x+7} + \frac{B}{(x+7)^2} + \frac{C}{(x+7)^3}$$
$$= \frac{A(x+7)^2 + B(x+7) + C}{(x+7)^3}$$
$$= \frac{x^2(A) + x(2\times 7A + B) + (7^2A + 7B + C)}{(x+7)^3}$$

Thus $A = 3\times 7$,

$3\times 7^2 = 2\times 7A + B$

So $B = 3\times 7^2 - 2\times 3\times 7^2 = -3\times 7^2$, and $7^2A + 7B + C = 7^3$

thus $\cancel{7^3} \times 3 - 3\times 7^3 + C = 7^3$

and $C = 7^3$.

Hence $\dfrac{x^3}{(x+7)^3} = 1 - \dfrac{7^3}{(x+7)^3} + \dfrac{3\times 7^2}{(x+7)^2} - \dfrac{3\times 7}{x+7}$

and $\displaystyle\int \frac{x^3}{(x+7)^3}\,dx = x + \frac{7^3}{2}\frac{1}{(x+7)^2} - 3\times 7^2 \frac{1}{x+7}$

$- 3\times 7 \ln|x+7| + C$

(2) (i) $\int_{-1}^{1} x^2 \arctan(x^3+1)\,dx$

$$t = x^3+1 \qquad x=1 \to t=2$$
$$dt = 3x^2\,dx \qquad x=-1 \to t=0$$

$= \frac{1}{3}\int_0^2 \arctan(t)\,dt.$

$$u = \arctan(t) \qquad u' = \frac{1}{1+t^2}$$
$$v' = 1 \qquad v = t$$

$= \frac{1}{3}\Big[t\arctan(t)\Big]_0^2 - \frac{1}{3}\int_0^2 \frac{t}{t^2+1}\,dt$

$= \frac{1}{3}\big(2\arctan(2)\big) - \frac{1}{3}\Big[\frac{1}{2}\ln(t^2+1)\Big]_0^2$

$= \frac{2}{3}\arctan(2) - \frac{1}{6}\ln(5)$

(since $\ln(1)=0$).

(ii) $\int_0^{1/2} \frac{3}{\sqrt{2x-x^2}}\,dx$

we do square completion:

$$\begin{aligned} -(x^2-2x) &= -\big((x^2-2x+1)-1\big) \\ &= -\big((x-1)^2-1\big) \\ &= 1-(x-1)^2. \end{aligned}$$

Thus $\int_0^{1/2} \frac{3}{\sqrt{2x-x^2}}\,dx$

$$= \int_0^{1/2} \frac{3}{\sqrt{1-(x-1)^2}}\,dx$$

$x-1 = \sin u$ $\quad \Rightarrow u = \arcsin(x-1)$.

$dx = \cos u\,du$

when $x = \frac{1}{2}$, $\sin u = -\frac{1}{2}$

so $u = -\frac{\pi}{6}$

when $x = 0$, $\sin u = -1$

so $u = -\frac{\pi}{2}$.

$$= \int_{-\frac{\pi}{2}}^{-\pi/6} \frac{3}{\sqrt{1-\sin^2 u}} \cos u\,du = \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} \frac{3\cos u}{\sqrt{\cos^2 u}}\,du$$

$$= \int_{-\frac{\pi}{2}}^{-\frac{\pi}{6}} 3\,du \quad \left(\text{since } \cos u \geq 0 \text{ on } \left[-\frac{\pi}{2}, -\frac{\pi}{6}\right]\right).$$

$$= \Big[3u\Big]_{-\pi/2}^{-\pi/6}$$

$$= -\frac{\pi}{2} + \frac{3\pi}{2} = \pi$$

(3)(i) $\displaystyle\int \frac{du}{1+u^4} = \int \frac{du}{(1+\sqrt{2}u+u^2)(1-\sqrt{2}u+u^2)}$.

both polynome are irreducible, so

$$\frac{1}{1+u^4} = \frac{Au+B}{1+\sqrt{2}u+u^2} + \frac{Cu+D}{1-\sqrt{2}u+u^2}$$

$$= \frac{(Au+B)(1-\sqrt{2}u+u^2) + (Cu+D)(1+\sqrt{2}u+u^2)}{1+u^4}$$

$$= \frac{u^3(A+C) + u^2(B-\sqrt{2}A+D+\sqrt{2}C) + u(-B\sqrt{2}+A+D\sqrt{2}+C) + (B+D)}{1+u^4}$$

Thus $\begin{cases} A+C=0 \\ B-\sqrt{2}A+D+\sqrt{2}C=0 \quad (1) \\ A-B\sqrt{2}+C+\sqrt{2}D=0 \quad (2) \\ B+D=1 \end{cases}$

so $A=-C$

$B=1-D$

thus (1) and (2) becomes

$\begin{cases} 1+2\sqrt{2}C=0 \\ 2\sqrt{2}D-\sqrt{2}=0 \end{cases}$

thus $\begin{cases} C=\frac{-1}{2\sqrt{2}} = \frac{-\sqrt{2}}{4} \\ D=+\frac{1}{2} \end{cases}$

and $A=+\frac{\sqrt{2}}{4}$, $B=\frac{1}{2}$.

So $\displaystyle\frac{1}{1+u^4} = \frac{\frac{1}{2}+\frac{\sqrt{2}}{4}u}{1+u\sqrt{2}+u^2} + \frac{\frac{1}{2}-\frac{\sqrt{2}}{4}u}{1-u\sqrt{2}+u^2}$

Since $\frac{d}{du}(u^2 + u\sqrt{2} + 1) = \sqrt{2} + 2u = \sqrt{2}(1 + \sqrt{2}u)$

we write

$$\frac{\frac{1}{2} + \frac{\sqrt{2}}{4}u}{1 + u\sqrt{2} + u^2} = \frac{1}{4}\left(\frac{\sqrt{2}u + 1}{1 + u\sqrt{2} + u^2}\right) + \frac{1/4}{1 + u\sqrt{2} + u^2}$$

Similarly we write

$$\frac{\frac{1}{2} - \frac{\sqrt{2}}{4}u}{1 - u\sqrt{2} + u^2} = \frac{-1}{4}\left(\frac{\sqrt{2}u - 1}{1 - u\sqrt{2} + u^2}\right) + \frac{1/4}{1 - u\sqrt{2} + u^2}$$

performing square completion:

$$1 + \sqrt{2}u + u^2 = \left(u^2 + 2\frac{\sqrt{2}}{2}u + \frac{2}{4}\right) + \frac{2}{4}$$
$$= \left(u + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}$$

Similarly,

$$1 - \sqrt{2}u + u^2 = \left(u - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}.$$

Then

$$\int \frac{du}{1 + u^4} = \frac{1}{4}\int \frac{\sqrt{2}u + 1}{u^2 + u\sqrt{2} + 1}\,du \;(*) + \frac{1}{4}\int \frac{du}{\left(u + \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} \;(***)$$
$$- \frac{1}{4}\int \frac{+\sqrt{2}u - 1}{u^2 - u\sqrt{2} + 1}\,du \;(**) + \frac{1}{4}\int \frac{du}{\left(u - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} \;(****)$$

$$(*) = \frac{1}{4\sqrt{2}}\int \frac{2u+\sqrt{2}}{u^2+\sqrt{2}u+1}\,du = \frac{1}{4\sqrt{2}}\ln\left(u^2+\sqrt{2}u+1\right) + C_1$$

$$(**) = \frac{-1}{4\sqrt{2}}\int \frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1}\,du = \frac{-1}{4\sqrt{2}}\ln\left(u^2-\sqrt{2}u+1\right) + C_2$$

$$\binom{**}{*} = \frac{1}{4}\int \frac{\frac{1}{\sqrt{2}}\sec^2 v\,dv}{\frac{1}{2}(\tan^2 v+1)} = \frac{1}{4}\times\frac{2}{\sqrt{2}}\int \frac{\sec^2 v}{\sec^2 v}\,dv$$

$u + \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\tan v$

$du = \frac{1}{\sqrt{2}}\sec^2 v\,dv$

$$= \frac{\sqrt{2}}{4}\int dv = \frac{\sqrt{2}}{4}v + C$$

$$= \frac{\sqrt{2}}{4}\arctan\left(u\sqrt{2}+1\right) + C_3$$

$$\binom{**}{**} = \frac{1}{4}\times\frac{2}{\sqrt{2}}\int\frac{\sec^2 v}{\sec^2 v}\,dv = \frac{\sqrt{2}}{4}\int dv = \frac{\sqrt{2}}{4}v + C$$

$u - \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\tan v$

$du = \frac{1}{\sqrt{2}}\sec^2 v\,dv$

$$= \frac{\sqrt{2}}{4}\arctan\left(u\sqrt{2}-1\right) + C_4$$

Thus
$$\int\frac{du}{1+u^4} = \frac{1}{4\sqrt{2}}\ln(u^2+\sqrt{2}u+1) - \frac{1}{4\sqrt{2}}\ln(u^2-\sqrt{2}u+1)$$
$$+ \frac{\sqrt{2}}{4}\arctan(u\sqrt{2}+1)$$
$$+ \frac{\sqrt{2}}{4}\arctan(u\sqrt{2}-1).$$
$$= \frac{1}{4\sqrt{2}}\ln\left(\frac{u^2+\sqrt{2}u+1}{u^2-\sqrt{2}u+1}\right) + \frac{\sqrt{2}}{4}\arctan(u\sqrt{2}+1)$$
$$+\frac{\sqrt{2}}{4}\arctan(u\sqrt{2}-1).$$

(3)(ii) $\int_0^{\infty} \frac{dx}{(1+x^2)\sqrt{x}}$ comparison test.

First note that there are problems at 0 and ∞. Thus we need to split the integral in two parts:

$$\int_0^{\infty} \frac{dx}{(1+x^2)\sqrt{x}} \to \int_1^{\infty} \frac{dx}{(1+x^2)\sqrt{x}} \quad (*)$$

$$\to \int_0^{1} \frac{dx}{(1+x^2)\sqrt{x}} \quad (**)$$

$(*)$: $\frac{1}{(1+x^2)\sqrt{x}} = \frac{1}{x^{1/2}+x^{5/2}} \underset{x\to\infty}{\sim} \frac{1}{x^{5/2}}$, whose integral at ∞ converges. Thus we expect $\int_1^{\infty} \frac{dx}{x^{1/2}+x^{5/2}}$ to converges.

if $x > 0$, $x^{5/2} + x^{1/2} \geq x^{5/2}$

thus $\frac{1}{x^{5/2}+x^{1/2}} \leq \frac{1}{x^{5/2}}$

thus $\int_1^{\infty} \frac{dx}{x^{5/2}+x^{1/2}} \leq \int_1^{\infty} \frac{dx}{x^{5/2}}$ converging since of the form $\int_1^{\infty} \frac{dx}{x^p}$, $p > 1$

$(**)$ here, $\frac{1}{x^{1/2}+x^{5/2}} \underset{x\to 0}{\sim} \frac{1}{x^{1/2}}$, and $\int_0^1 \frac{dx}{\sqrt{x}}$ converges.

(14)

we need to major $\frac{1}{x^{1/2}+x^{5/2}}$:

$x^{1/2}+x^{5/2} \geqslant x^{1/2} \quad (x \geqslant 0)$

So $\frac{1}{x^{1/2}+x^{5/2}} \leq \frac{1}{x^{1/2}}$

thus $\int_0^1 \frac{dx}{x^{1/2}+x^{5/2}} \leq \int_0^1 \frac{dx}{x^{1/2}}$, converging since of the form $\int_0^1 \frac{1}{x^p}\,dx$, $p<1$.

thus since both $(*)$ and $(**)$ converge, so does $\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}}$.

(15)

(3)(iii) $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}} = \int_0^\infty \frac{2u\,du}{(1+u^4)\,u} = 2\int_0^\infty \frac{du}{1+u^4}$

$u = \sqrt{x} \rightarrow x = u^2$ | when $x=0$, $u=0$

$\rightarrow 2u\,du = dx$ | $x \to \infty$, $u \to \infty$

$$= \underbrace{2\lim_{t\to 0}\int_t^1 \frac{du}{1+u^4}}_{(*)} + \underbrace{2\lim_{t\to\infty}\int_1^\infty \frac{du}{1+u^2}}_{(**)}.$$

- $\displaystyle\int_t^1 \frac{du}{1+u^2} = \overbrace{\frac{1}{4\sqrt{2}}\ln\left(\frac{1+\sqrt{2}}{1-\sqrt{2}}\right) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}+1) + \frac{\sqrt{2}}{4}\arctan(\sqrt{2}-1)}^{\text{call it } F(1)}$

$$- \frac{1}{4\sqrt{2}}\underbrace{\ln\left(\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right)}_{\downarrow t\to 0 \\ \ln(1)=0} - \frac{\sqrt{2}}{4}\underbrace{\arctan(\sqrt{2}t+1)}_{\downarrow t\to 0 \\ \arctan(1) = \frac{\pi}{4}} + \frac{\sqrt{2}}{4}\underbrace{\arctan(\sqrt{2}t-1)}_{\downarrow t\to 0 \\ \arctan(-1) = -\frac{\pi}{4}}$$

So $\displaystyle\lim_{t\to 0}\int_t^1 \frac{du}{1+u^2} = F(1)$.

- $\displaystyle\int_1^t \frac{du}{1+u^2} = \underbrace{\frac{1}{4\sqrt{2}}\ln\Bigg(\underbrace{\frac{t^2+\sqrt{2}t+1}{t^2+\sqrt{2}t-1}}_{\to 1 \text{ as } t\to\infty}\Bigg)}_{\to 0 \text{ as } t\to\infty} + \frac{\sqrt{2}}{4}\underbrace{\arctan(\sqrt{2}t+1)}_{\downarrow t\to\infty \\ \frac{\pi}{2}} + \frac{\sqrt{2}}{4}\underbrace{\arctan(\sqrt{2}t-1)}_{\downarrow t\to\infty \\ \frac{\pi}{2}} - F(1).$

So

$$\lim_{t\to\infty}\int_1^t \frac{du}{1+u^2} = \frac{\sqrt{2}}{4}\left(\frac{\pi}{2}\right) + \frac{\sqrt{2}}{4}\left(\frac{\pi}{2}\right) - F(1) = \frac{\pi}{4}\sqrt{2} - F(1)$$

thus $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)\sqrt{x}} = \frac{\pi\sqrt{2}}{4} - F(1) + F(1)$

$$= \frac{\pi\sqrt{2}}{4}.$$

(4)(i) $\int_1^4 \frac{dx}{(x-2)^{2/3}}$: improper integral, at $x=2$.

So $\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$.

$$\int \frac{dx}{(x-2)^{2/3}} = \int \frac{du}{u^{2/3}} = \int u^{-2/3}\,du = 3u^{1/3} + C$$

$u = x-2$
$du = dx$

$$= 3(x-2)^{1/3} + C.$$

$$\int_1^2 \frac{dx}{(x-2)^{2/3}} = \lim_{t\to 2^-} \int_1^t \frac{dx}{(x-2)^{2/3}} = \lim_{t\to 2^-} \left[3(x-2)^{1/3}\right]_1^t$$

$$= \lim_{t\to 2^-} \Big(\underbrace{3(t-2)^{1/3}}_{\downarrow t\to 2^- \; 0} - 3(1-2)^{1/3}\Big)$$

$$= -3(1-2)^{1/3} = 3$$

$$\int_2^4 \frac{dx}{(x-2)^{2/3}} = \lim_{t\to 2^+} \int_t^4 \frac{dx}{(x-2)^{2/3}} = \lim_{t\to 2^+} \Big(3(4-2)^{1/3} - \underbrace{3(t-2)^{1/3}}_{\downarrow t\to 2^+ \; 0}\Big)$$

$$= 3\times 2^{1/3} = 3\sqrt[3]{2}$$

Thus $\int_1^4 \frac{dx}{(x-2)^{2/3}}$ converges to $3(1+\sqrt[3]{2})$

(4)(ii) $\int_{\pi/6}^{\pi/2} \frac{1}{\tan x}\,dx$: improper at $\frac{\pi}{2}$.

$$\int \frac{dx}{\tan x} = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} = \ln|u|$$

$u = \sin x$

$du = \cos x\,dx$

$$= \ln|\sin x|.$$

Thus $$\int_{\pi/6}^{\pi/2} = \lim_{t\to\frac{\pi}{2}} \Big(\underbrace{\ln|\underbrace{\sin t}_{\to 1,\ t\to\frac{\pi}{2}}|}_{\to \ln(1) = 0,\ t\to\frac{\pi}{2}} - \ln|\sin\tfrac{\pi}{6}|\Big)$$

$$= -\ln|\sin\tfrac{\pi}{6}| = -\ln|\tfrac{1}{2}| = \ln(2)$$

(iii) $\int_2^{\infty} \frac{e^{-x}}{3+e^{-x}}\,dx$ improper at ∞

$u = e^{-x}$

$du = -e^{-x}dx$

$$\int_2^t \frac{e^{-x}}{3+e^{-x}}\,dx = -\int_{x=2}^{t} \frac{du}{3+u} = -\Big[\ln|3+u|\Big]_{x=2}^{x=t}$$

$$= -\Big[\ln|3+e^{-x}|\Big]_2^t$$

$$= -\ln|3+e^{-t}| + \ln|3+e^{-2}|$$

as $t\to\infty$, $e^{-t}\to 0$ so

$$\lim_{t\to\infty}\int_2^t \frac{e^{-x}}{3+e^{-x}}\,dx = -\ln(3)+\ln(3+e^{-2})$$

$$= \int_2^\infty \frac{e^{-x}}{3+e^{-x}}\,dx$$

(iv) $\int_3^\infty \frac{dx}{x\sqrt[3]{\ln x}}$: improper at ∞.

$$\int_3^t \frac{dx}{x\sqrt[3]{\ln x}} = \int_{x=3}^{x=t} \frac{du}{\sqrt[3]{u}} = \int_{x=3}^{x=t} u^{-1/3}\,du$$

$u = \ln x$

$du = \frac{dx}{x}$

$$= \left[\frac{3}{2}u^{2/3}\right]_{x=3}^{x=t}$$

$$= \left[\frac{3}{2}(\ln x)^{2/3}\right]_3^t$$

$$= \underbrace{\frac{3}{2}(\ln t)^{2/3}}_{\downarrow t\to\infty \atop \infty} - \frac{3}{2}(\ln 3)^{2/3}$$

So $\lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt[3]{\ln x}} = \infty$: the integral diverges

(5) (i) $\int_1^\infty \frac{dx}{\sqrt{x}+e^{2x}}$.

at x goes to ∞, $\sqrt{x}+e^{2x} \sim e^{2x}$.
So expect convergence for the integral.

$\sqrt{x}+e^{2x} \geqslant e^{2x}$, so $\frac{1}{\sqrt{x}+e^{2x}} \leq \frac{1}{e^{2x}} = e^{-2x}$.

$\int_1^\infty e^{-2x}\,dx$ converges, so does

$$\int_1^\infty \frac{dx}{\sqrt{x}+e^{2x}}.$$

(ii) $\int_0^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$.

problem at 0 and ∞.

consider $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ first.

at $x \to \infty$, $1+\sqrt{x} \sim \sqrt{x}$, so $\sqrt{1+\sqrt{x}} \sim \sqrt[4]{x}$,

so $\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \sim \frac{1}{\sqrt[4]{x}} = \frac{1}{x^{1/4}}$.

this does not have a converging integral at ∞.

let's minor $\dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}$:

$\sqrt{1+\sqrt{x}} \geqslant \sqrt{\sqrt{x}}$ so $\dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} \geqslant \dfrac{\sqrt{\sqrt{x}}}{\sqrt{x}} = \dfrac{1}{\sqrt[4]{x}} = \dfrac{1}{x^{1/4}}$.

Thus $\underbrace{\int_1^\infty \frac{dx}{x^{1/4}}}_{} \leqslant \int_1^\infty \dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$

diverges,
since of the form $\int_1^\infty \frac{1}{x^p}\,dx$, $p<1$

Thus $\int_1^\infty \dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$ diverges,

and so does $\int_0^\infty \dfrac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,dx$.

(iii) $\int_0^1 \dfrac{dx}{\sqrt[5]{x^4+x^8}}$: improper at 0.

$x^4+x^8 \underset{x\to 0}{\sim} x^4$: so $\sqrt[5]{x^4+x^8} \underset{x\to 0}{\sim} x^{4/5}$.

$\int_0^1 \dfrac{dx}{x^{4/5}}$ converges, so we suspect the original one to converge.

Since $x^4 \leqslant x^4+x^8$, $\sqrt[5]{x^4} \leqslant \sqrt[5]{x^4+x^8}$

so $\dfrac{1}{\sqrt[5]{x^4}} \geqslant \dfrac{1}{\sqrt[5]{x^4+x^8}}$.

Thus since $\int_0^1 \dfrac{1}{\sqrt[5]{x^4}}\,dx$ converges, so does $\int_0^1 \dfrac{dx}{\sqrt[5]{x^4+x^8}}$.

(w) $\displaystyle\int_0^{\pi/2} \frac{dx}{x^3\cos^2 x}$ problem at 0 and $\frac{\pi}{2}$.

at 0, $\cos^2 x \underset{x\to 0}{\sim} 1$ so $\dfrac{1}{x^3\cos^2 x} \sim \dfrac{1}{x^3}$.

Since $\displaystyle\int_0^1 \frac{1}{x^3}$ diverges, we suspect the original integral to do so as well.

$\cos^2 x \leq 1$ on $[0, \frac{\pi}{2}]$,

So $x^3\cos^2 x \leq x^3$

and $\dfrac{1}{x^3\cos^2 x} \geq \dfrac{1}{x^3}$

Thus since $\displaystyle\int_0^1 \frac{dx}{x^3}$ diverges,

$\displaystyle\int_0^1 \frac{dx}{x^3\cos^2 x}$ diverges.

Thus $\displaystyle\int_0^{\pi/2} \frac{dx}{x^3\cos^2 x}$ diverges